KMS states and fixed-point theory

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Directed graphs

A *directed graph* is a quadruple $E = (E^0, E^1, r, s)$, where E^0 and E^1 are countable sets, and r, s are functions from E^1 to E^0.

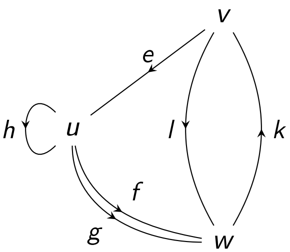
- We call the elements of E^0 *vertices* and think of them as points.
- We call the elements of E^1 *edges* and think of them as arrows pointing from one vertex to another.
- The edge $e \in E^1$ points from $s(e) \in E^0$ to $r(e) \in E^0$.

$$s(e) \xrightarrow{e} r(e)$$

A *path* in E is a word $e_1 e_2 \cdots e_n$ of edges such that $s(e_i) = r(e_{i+1})$.

An example.

One example of a directed graph is:



An example of a path in this graph is *lkfhe*.

For today,

- both E^0 and E^1 are finite and nonempty; and
- Strongly connected: each vE^*w is nonempty.

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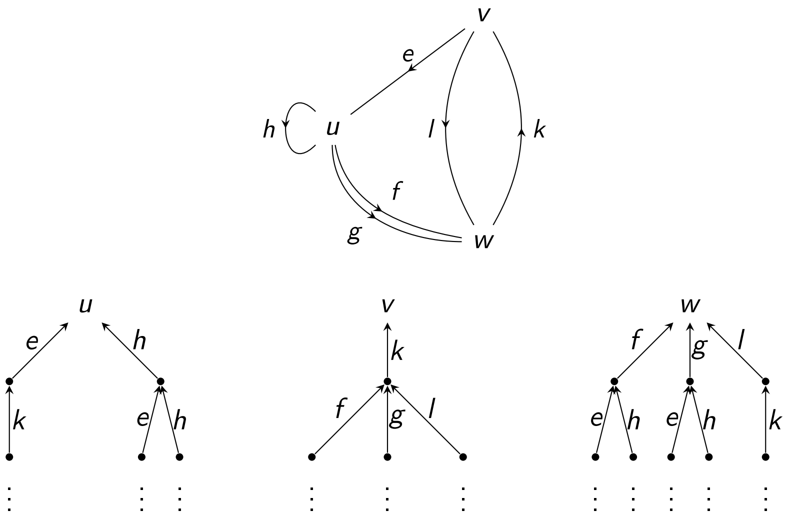
Paths in graphs

- A *path of length* $n \geq 1$ in a graph E is a word $\lambda = \lambda_1 \dots \lambda_n$ with each $\lambda_i \in E^1$ such that $s(\lambda_i) = r(\lambda_{i+1})$.
- An *infinite path* in E is a word $x = x_1 x_2 x_3 \dots$ where each $x_i \in E^1$ and each $s(x_i) = r(x_{i+1})$.
- E^∞ is the space of all infinite paths. Give it the topology generated by the sets λE^∞ (these are then compact and open, and the topology is Hausdorff).

The path-space of a graph

The path-space of a graph E forms a forrest F_E:



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Partial automorphisms

[LRRW]: A *partial isomorphism* of F_E is a triple (v, γ, w) where $v, w \in E^0$, and $\gamma : wE^* \to vE^*$ is a length-preserving bijection such that $\gamma(\mu e) \in \gamma(\mu)E^1$ for all paths μ and edges e.

The collection $\mathrm{PIso}(F_E)$ of partial isomorphisms of E is a groupoid with unit space E^0:

- (v, γ, w) and (x, η, y) compose, to give $(v, \gamma \circ \eta, y)$ if $w = x$.
- $r(v, \gamma, w) = v$ and $s(v, \gamma, w) = w$.
- $(v, \gamma, w)^{-1} = (w, \gamma^{-1}, v)$.

Write γ for (v, γ, w) and write $r(\gamma) = v$ and $s(\gamma) = w$.

Eg: $E^0 = \{v\}$, $E^1 = \{1, \ldots, n\}$, then $\mathrm{PIso}(F_E) =$ automorphism group of rooted n-ary tree.

Self-similar groupoids

[LRRW] A *self-similar groupoid* Γ is a subgroupoid of $\mathrm{PIso}(F_E)$ for some graph E, with the property that for each $\gamma \in \Gamma$ and $\mu \in wE^*$, there is a (unique) $\gamma|_\mu \in \Gamma$ such that $\gamma \cdot (\mu\nu) = (\gamma \cdot \mu)(\gamma|_\mu \cdot \nu)$ for all ν.

So if E has just one vertex, then this is just the usual notion of a self-similar group.

For example, if E has one vertex and two edges $0, 1$, then the odometer subgroup of $\mathrm{Aut}\{0,1\}^*$ with generator g given by $g \cdot 0 = 1$, $g \cdot 1 = 0$, $g|_0 = e$ and $g|_1 = g$ is a self-similar group isomorphic to $\mathbb{Z}$.

Toeplitz algebras of graphs

C^*-algebra: closed *-algebra of $B(H)$. To model a directed graph in a C^*-algebra:

- Assign orthogonal subspaces H_v to vertices v;
- Write p_v for the orthogonal projection onto H_v;
- Assign an operator s_e to each edge e so that
  - s_e is isometric from $H_{s(e)}$ to a subspace of $H_{r(e)}$,
  - s_e is zero on H_v if $v \neq s(e)$, and
  - if $e \neq f$ then $s_e H_{s(e)} \perp s_f H_{s(f)}$.

$\mathcal{T}C^*(E)$ is the universal C^*-algebra generated by elements p_v, s_e such that $p_v^* p_w = \delta_{v,w} p_v$, each $s_e^* s_e = p_{s(e)}$, and each $\sum_{r(e)=v} s_e s_e^*$ is a projection dominated by p_v.

The universal property gives an action $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{T}C^*(E))$ such that $\alpha_t(s_e) = e^{it} s_e$ and $\alpha_t(p_v) = p_v$.

KMS states on $\mathcal{T}C^*(E)$

Reminder: If A is a C^*-algebra, a *state* of a is a linear map $\phi : A \to \mathbb{C}$ of norm 1 such that $\phi(a^*a) \geq 0$ for all a.

If $\alpha : \mathbb{R} \to \mathrm{Aut}(A)$ is an action on a C^*-algebra, and $\beta > 0$, then a state $\phi : A \to \mathbb{C}$ is KMS_β if $\phi(ab) = \phi(b\alpha_{i\beta}(a))$ whenever this makes sense.

Theorem (aHLRS). If E is a strongly-connected finite directed graph and $\beta > 0$ then there are KMS_β states of $\mathcal{T}C^*(E)$ if and only if β is larger than the logarithm of the spectral radius ρ of the adjacency matrix A_E. For $\beta > \log \rho$, KMS_β-states $\leftrightarrow$ probability measures on E^0; at $\beta = \log \rho$ there is a unique KMS state, given on the p_v by the entries of the Perron–Frobenius eigenvector m^E of A_E.

Idea of proof

Write $s_\mu = s_{\mu_1} s_{\mu_2} \cdots s_{\mu_n}$ for a path $\mu = \mu_1 \cdots \mu_n$.

Relations force $\mathcal{T}C^*(E) = \overline{\operatorname{span}}\{s_\mu s_\nu^* : s(\mu) = s(\nu)\}$, and if $|\mu| = |\nu|$ then $s_\mu^* s_\mu = \delta_{\mu,\nu} p_{s(\mu)}$.

If $|\mu| \neq |\nu|$ and ϕ is KMS$_\beta$, then $\phi(s_\mu s_\nu^*) = \phi(s_\nu^* \alpha_{i\beta}(s_\mu)) = \phi(\alpha_{i\beta}(s_\mu)\alpha_{i\beta}(s_\nu^*)) = e^{-\beta(|\mu|-|\nu|)}\phi(s_\mu s_\nu^*)$. Hence $\phi(s_\mu s_\nu^*) = 0$.

If $|\mu| = |\nu|$ then $\phi(s_\mu s_\nu^*) = \phi(s_\nu^* \alpha_{i\beta}(s_\mu)) = \delta_{\mu,\nu} e^{-\beta|\mu|} p_{s(\mu)}$.

Also, each $\phi(p_v) \geq \sum_{r(e)=v} \phi(s_e s_e^*) = e^{-\beta} \sum_{r(e)=v} s_e^* s_e = e^{-\beta} \sum_w A_E(v,w)\phi(p_w)$.

So KMS$_\beta$ states $\leftrightarrow$ probability measures m with $A_E m \leq e^\beta m$. Perron–Frobenius kicks in.

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A critical observation

How to find measures m with $A_E m \leq e^{-\beta} m$ (called *subinvariant*)?

If $e^{\beta} > \rho$ then $\sum_n e^{-n\beta} A_E^n$ converges.

Follows that $\sum_n e^{-n\beta} A_E^n m$ converges for any measure m.

$$A_E\Big(\sum_{n\geq 0} e^{-n\beta} A_E^n m\Big) = \sum_{n\geq 0} e^{-n\beta} A_E^{n+1} m$$
$$= e^{\beta} \sum_{n\geq 1} e^{-n\beta} A_E^n m \leq e^{\beta} \sum_{n\geq 0} e^{-n\beta} A_E^n m.$$

So, modulo scaling, get subinvariant measure via
$\chi_\beta(m)(v) = \sum_{\mu\in E^* v} e^{-\beta|\mu|} m(r(\mu))$.

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Toeplitz algebras of self-similar groupoids [LRRW]

Given: strongly-connected finite directed graph E, and self-similar groupoid $\Gamma \subseteq \mathrm{PIso}(F_E)$.

Define $\mathcal{T}(E, \Gamma)$ to be universal C^*-algebra generated by

- p_v and s_e as before, *and*
- $\{u_\gamma : \gamma \in \Gamma\}$ such that
  - $u_{\gamma^{-1}} = u_\gamma^*$,
  - $u_\gamma^* u_\gamma = p_{s(\gamma)}$ (hence $u_\gamma u_\gamma^* = p_{r(\gamma)}$),
  - $u_\gamma s_\mu = s_{\gamma\cdot\mu} u_{\gamma|_\mu}$ when $s(\gamma) = r(\mu)$.

So, roughly, a copy of $\mathcal{T}C^*(E)$ and a "unitary representation" of Γ that play nicely together. Have $\mathcal{T}C^*(E, \Gamma) = \overline{\mathrm{span}}\{s_\mu u_\gamma s_\nu^*\}$.

The universal property gives an action $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{T}C^*(E, \Gamma))$ such that $\alpha_t(s_e) = e^{it} s_e$, $\alpha_t(p_v) = p_v$, and $\alpha_t(u_\gamma) = u_\gamma$.

KMS states on $\mathcal{T}C^*(E,\Gamma)$

Suppose that ϕ is KMS_β for $(\mathcal{T}C^*(E,\Gamma),\alpha)$.

Then in particular $\phi|_{C^*(\{s_e,p_v\})}$ is KMS_β for $\mathcal{T}C^*(E)$. We know about these. Also, $\tau(s_\mu u_\gamma s_\nu^*) = 0$ if $\mu \neq \nu$ as before.

But also $\phi(u_\gamma u_\eta) = \phi(u_\eta \alpha_{-i\beta}(u_\gamma)) = \phi(u_\eta u_\gamma)$ because $\alpha_t(u_\gamma) = u_\gamma$.

So $\phi|_{C^*(\Gamma)}$ is a trace. Hence supported on $C^*(\{\gamma : s(\gamma) = r(\gamma)\})$, a sum of isotropy-group C^*-algebras.

If $s(\gamma) = r(\gamma) = v$ and $\gamma \cdot \mu \neq \mu$, then

$$\phi(u_\gamma s_\mu s_\mu^*) = \phi(s_{\gamma\cdot\mu} u_{\gamma|_\mu} s_\mu^*) = e^{-\beta}\phi(u_{\gamma|_\mu} s_e^* s_{\gamma\cdot\mu}) = 0.$$

So ϕ “sees” how much of $s(\gamma)E^*$ is fixed by γ.

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KMS states on $\mathcal{T}C^*(E,\Gamma)$

Theorem. [LRRW] Let E be a strongly connected directed graph and let $\Gamma \subseteq \mathrm{PIso}(F_E)$ be a self-similar groupoid. Let τ be a trace of $C^*(\Gamma)$. Fix $\beta > \log \rho$. The series

$$Z(\beta,\tau) := \sum_{k=0}^{\infty} e^{-k\beta} \sum_{\mu \in E^k} \tau(u_{s(\mu)})$$

converges to a positive real number, and there is isomorphism $\tau \mapsto \Psi_{\beta,\tau}$ from $\mathrm{Tr}(C^*(\Gamma))$ to $\mathrm{KMS}_\beta(\mathcal{T}C^*(E,\Gamma))$ such that

$$\Psi_{\beta,\tau}(s_\mu u_\gamma s_\nu^*) = \delta_{\mu,\nu} e^{-\beta|\mu|} Z(\beta,\tau)^{-1} \sum_{k=0}^{\infty} \Bigg(\sum_{\lambda \in s(\mu)E^k, \gamma\cdot\lambda=\lambda} \tau(u_{\gamma|_\lambda}) \Bigg).$$

Under mild technical hypothesis, there is a unique $\mathrm{KMS}_{\log\rho}$-state ψ_c, and

$$\psi_c(u_\gamma) = \lim_n \rho^{-n} \sum_v \big|\{\mu \in E^n : \gamma\cdot\mu = \mu \text{ and } \gamma|_\mu = v\}\big| m_v^E.$$

Preferred traces

So self-similar action yields "preferred trace" τ_c on $C^*(\Gamma)$—roughly $\tau_c(u_\gamma)$ is the measure of $\{x \in E^\infty : \gamma \cdot x = x\}$.

Example [LRRW]: for the Basilica group acting self-similarly on the binary tree by

$$a \cdot 0w = 1(b \cdot w), \qquad a \cdot 1w = 0w,$$
$$b \cdot 0w = 0(a \cdot w), \qquad b \cdot 1w = 1w,$$

we have $\tau_c(b) = \tau_c(b^{-1}) = \frac{1}{2}$, and $\tau(\{a, a^{-1}, ab^{-1}, ba^{-1}\}) = 0$ (other values are determined by these).

Preferred traces

Example [LRRW]: for the Grigorchuk group with

$$\begin{aligned} a \cdot 0w &= 1w, & a \cdot 1w &= 0w, \\ b \cdot 0w &= 0(a \cdot w), & b \cdot 1w &= 1(c \cdot w), \\ c \cdot 0w &= 0(a \cdot w), & c \cdot 1w &= 1(d \cdot w), \\ d \cdot 0w &= 0w, & d \cdot 1w &= 1(b \cdot w), \end{aligned}$$

we have $\tau_c(u_a) = 0$, $\tau_c(u_b) = \frac{1}{7}$, $\tau_c(u_c) = \frac{2}{7}$, and $\tau_c(u_d) = \frac{4}{7}$.

Key observation (again)

For graphs, a KMS state of $\mathcal{T}C^*(E)$ came from a subivariant measure on E^0, and the graph determined a map from arbitrary measures to subinvariant measures.

For self-similar groupoids, the map $\tau \mapsto \Psi_{\beta,\tau}$ obtains a KMS_β-state from an arbitrary trace.

But then $\Psi_{\beta,\tau}|_{C^*(\Gamma)}$ is another trace.

That is, the self-similar action hands us a self-mapping χ_β of $\mathrm{Tr}(C^*(\Gamma))$; namely, $\chi_\beta(\tau) = \Psi_{\beta,\tau}|_{C^*(\Gamma)}$.

So what are the fixed points for this self-mapping χ_β?

A fixed-point result

Theorem. [CS] Let E be a strongly connected directed graph and let $\Gamma \subseteq \mathrm{PIso}(F_E)$ be a self-similar groupoid. Under the same technical assumption appearing in the [LRRW] theorem, for any $\beta > \log \rho$, the map $\chi_\beta : \mathrm{Tr}(C^*(\Gamma)) \to \mathrm{Tr}(C^*(\Gamma))$ has a unique fixed point τ_c. This τ_c is equal to the restriction of ψ_c to $C^*(\Gamma)$.

Outline of proof

Lemma 1. The map χ_β is weak*-continuous. Hence any limit-point of the form $\theta = \lim_n \chi_\beta^n(\tau)$ is a fixed point.

Lemma 2. Let $N(\beta,\tau) := e^\beta(1 - Z(\beta,\tau)^{-1})$. If τ is a fixed point for χ_β, then

$$N(\beta,\tau)^n \tau(u_\gamma) = \sum_{\mu \in E^n, \gamma\cdot\mu = \mu} \tau(u_{\gamma|_\mu}).$$

For *any* τ satisfying the above, $(\tau(u_v))_{v \in E^0} = m^E$, and $N(\beta,\tau) = \rho$.

Lemma 3. The matrix $A_{vN} = (I - e^{-\beta} A_E)^{-1}$ is primitive, and $\chi_\beta^n(\tau)|_{\mathbb{C}E^0} = \frac{1}{\|A_{vN}(\tau|_{\mathbb{C}E^0})\|} A_{vN}(\tau|_{\mathbb{C}E^0})$.

Outline of proof

Corollary 4. For any τ, we have $\chi_\beta^n(\tau)|_{\mathbb{C}^{E^0}} \to m^E$ exponentially quickly (in 1-norm).

The point here is that we can apply Perron–Frobenius theory to the matrix A_{vN}, and then this is a standard result.

Roughly speaking, this says that to analyse the sequence $\chi_\beta^n(\tau)$ for an arbitrary state τ, it suffices to do this for τ satisfying $\big(\tau(u_v)\big)_{v\in E^0} = m^E$.

Outline of proof

Theorem 6. Suppose that θ is a trace of $C^*(\Gamma)$ that satisfies

$$N(\beta,\theta)^n\theta(u_\gamma) = \sum_{\mu\in E^n,\gamma\cdot\mu=\mu} \theta(u_{\gamma|_\mu}) \qquad (*)$$

for all γ. Then $\lim_n \chi_\beta^n(\tau) = \theta$ for any trace τ.

This is where some analysis and the technical hypothesis from [LRRW] come into play. The analysis, like that of [LRRW] hinges on showing that for any nontrivial γ there are "not too many" paths μ such that $\gamma\cdot\mu = \mu$ but $\gamma|_\mu \neq s(\mu)$. We use this to find constants $0 < \lambda < 1$ and $K, D > 0$ that we can inductively demonstrate satisfy $|\chi_\beta^n(\tau)(u_g) - \theta(u_g)| < (nK + D)K\lambda^{n-1}$ for all n. then an $\varepsilon/3$-argument establishes the result because the u_g span a dense subspace of $C^*(\Gamma)$.

To finish off the proof, we use our earlier results to see that if ϕ_c is the unique $\mathrm{KMS}_{\log\rho}$ state from [LRRW], then $\theta = \phi_c|_{C^*(\Gamma)}$ satisfies

$$N(\beta,\theta)^n\theta(u_\gamma) = \sum_{\mu\in E^n,\gamma\cdot\mu=\mu} \theta(u_{\gamma|_\mu}) : \qquad (*)$$

We know from the graph algebra theorem that $\big(\theta(p_v)\big)_{v\in E^0} = m^E$, and then the definition of $N(\beta,\theta)$ shows that it is precisely ρ.

From [LRRW], $\phi_c(p_v) = \sum_{r(e)=v}\phi_c(s_e s_e^*)$ for each v. So

$$\begin{aligned}\phi_c(u_\gamma) &= \sum_e \phi_c(u_\gamma s_e s_e^*) = \sum_e \phi(s_{\gamma\cdot e} u_{\gamma|_e} s_e^*)\\ &= \sum_e \rho^{-1}\theta(u_{\gamma|_e} s_e^* s_{\gamma\cdot e}) = \rho^{-1}\sum_{\gamma\cdot e=e}\theta(u_{\gamma|_e}),\end{aligned}$$

which is (*) for $n=1$, and induction does the rest.