KMS states of C*-dynamical systems: an introduction and three examples

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C*-dynamical systems and states

A C*-dynamical system is a pair (A, σ) with

- A a C*-algebra (self adjoint elements = *observables*)
- $\sigma : \mathbb{R} \to \mathrm{Aut}(A)$ (the *dynamics* or time evolution on A)

(i.e. $\sigma_0 = \mathrm{id}$, $\sigma_s \circ \sigma_t = \sigma_{s+t}$ and $t \mapsto \sigma_t(a)$ is norm continuous)

A *state* of A is a linear functional $\varphi : A \to \mathbb{C}$ such that

$$\varphi(a^* a) \geqslant 0 \quad \text{and} \quad \|\varphi\| = \varphi(1) = 1$$

$\varphi(\sigma_t(a))$ is the *expectation value* of the observable $a \in A^{sa}$ corresponding to the state φ at time $t \in \mathbb{R}$.

Basic facts

1. The states of a commutative C*-algebra $A = C_0(\Omega_A)$ are in bijection with the probability measures on its spectrum Ω_A.

$$\varphi_\mu(f) = \int_{\Omega_A} f \, d\mu$$

2. If A is a C*-subalgebra of $B(\mathcal{H})$ for a Hilbert space $\mathcal{H}$, each unit vector $\xi \in \mathcal{H}$ gives rise to a state ω_ξ given by

$$\omega_\xi(a) := \langle a\xi, \xi \rangle.$$

   These are not all states, but...

3. **GNS construction:** for each state φ of A there is a Hilbert space $\mathcal{H}_\varphi$, a representation $\pi_\varphi : A \to B(\mathcal{H}_\varphi)$, and a cyclic unit vector $\xi_\varphi \in \mathcal{H}_\varphi$ such that

$$\varphi(a) = \langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi \rangle.$$

Example (finite quantum systems)

cf. [Hugenholtz, *C*-algebras and statistical mechanics* Proc. Symp. Pure Math. **38** (1981)]

- $A = \mathrm{Mat}_n(\mathbb{C})$, so the observables are the selfadjoint $n \times n$ matrices.
- Every time evolution σ on $\mathrm{Mat}_n(\mathbb{C})$ arises as $\sigma_t(a) = e^{itH} a e^{-itH}$ with H a selfadjoint matrix (a *Hamiltonian*) which is determined up to an additive constant, and is usually normalized so that its smallest eigenvalue is 0.
- There is a 1 to 1 correspondence between states φ of $\mathrm{Mat}_n(\mathbb{C})$ and *density matrices* Q_φ such that $\varphi(a) = \mathrm{Tr}(aQ_\varphi)$. ($Q \in \mathrm{Mat}_n(\mathbb{C})$ is a density matrix iff $Q \geqslant 0$ and $\mathrm{Tr}\, Q = 1$)
- φ is pure (i.e. extremal in the state space) iff Q_φ is a rank-one projection.

Example (finite quantum systems)

- The stationary (i.e. σ-invariant) states are those for which

$$\mathrm{Tr}(e^{itH} a e^{-itH} Q) = \mathrm{Tr}(aQ) \qquad \forall a \in \mathrm{Mat}_n(\mathbb{C});$$

  by the trace property $\mathrm{Tr}(a e^{-itH} Q e^{itH}) = \mathrm{Tr}(aQ)$ hence Q is stationary iff it commutes with e^{itH} and thus with H.

- Extremal stationary states are pure; their density matrices are the projections onto the eigenspaces of H.

- The *von Neumann entropy* of a state $\varphi = \mathrm{Tr}(\cdot\, Q_\varphi)$ is

$$S(\varphi) := -\mathrm{Tr}(Q_\varphi \log Q_\varphi).$$

  $S(\varphi) = 0$ (minimal) when φ is pure, and $S(\varphi) = \log n$ (maximal) when φ is the normalized trace. *"A pure state has maximal information, and the normalized trace has minimal information."*

Example (finite quantum systems)

Definition

The *free energy* of the state φ of $\mathrm{Mat}_n(\mathbb{C})$ with Hamiltonian H at inverse temperature $\beta = 1/T$ is $\quad F(\varphi) := -S(\varphi) + \beta\varphi(H)$,

The free energy (for fixed β and H) is minimized at a unique state:

Proposition (thermodynamic inequality)

1) $\quad F(\varphi) := -S(\varphi) + \beta\varphi(H) \geqslant -\log \mathrm{Tr}(e^{-\beta H})$;

2) equality holds if and only if φ is the **Gibbs state** φ_G*, having density matrix* $Q_G = \frac{1}{\mathrm{Tr}(e^{-\beta H})} e^{-\beta H}$.

For a proof see e.g. the appendix in [Hugenholtz, *C*-algebras and statistical mechanics*].

Example (finite quantum systems)

Proposition

φ_G is the unique state on $\mathrm{Mat}_n(\mathbb{C})$ such that

$$\varphi(ab) = \varphi(b\sigma_{i\beta}(a)) \qquad \textit{i.e.} \qquad \mathrm{Tr}(abQ) = \mathrm{Tr}(be^{-\beta H}ae^{\beta H}Q)$$

for all $a, b \in \mathrm{Mat}_n(\mathbb{C})$, where $\sigma_{i\beta}(a) = e^{-\beta H}ae^{\beta H}$.

Proof: Exercise. The key is to show Gibbs density is the only density that satisfies the above condition for every a, b in $\mathrm{Mat}_n(\mathbb{C})$ (an interesting exercise in linear algebra).

This is the KMS condition; it characterizes equilibrium for finite systems, and is the definition of equilibrium state at inverse temperature β for general C*-algebraic systems $(\mathfrak{A}, \sigma)$.

the KMS condition

Definition (Haag-Hugenholtz-Winnink, 1967)

A state φ on A satisfies the Kubo-Martin-Schwinger (KMS) condition with respect to σ at inverse temperature $\beta \neq 0$ (φ is a σ- KMS_β state), if

$$\varphi(ab) = \varphi(b\sigma_{i\beta}(a))$$

for all σ-analytic elements $a, b \in A$.

Remark: This is a tracelike condition, twisted by σ along ‘imaginary time’.

Analytic elements

Let (A, σ) be a C*-dynamical system; $a \in A$ is called *σ-analytic if the map* $\mathbb{R} \ni t \mapsto \sigma_t(a) \in A$ *extends to an A-valued entire function* $z \mapsto \sigma_z(a) \in A$*. Equivalently,* $t \mapsto f(\sigma_t(a))$ *extends to a complex valued entire function for every bounded linear functional f on A.*

Fact: The σ-analytic elements form a dense *-subalgebra of A.

For $a \in A$ the element

$$x_n := \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} \sigma_t(x) \exp(-nt^2)\, dt$$

is analytic for σ because

$$z \mapsto \sigma_z(x_n) := \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} \sigma_t(x) \exp(-n(t-z)^2)\, dt$$

is entire and $\|x_n - x\| \to 0$.

Sample property: KMS $\implies$ stationary

Proposition

If $\beta \neq 0$ and φ is a KMS$_\beta$ state, then φ is σ-invariant.

Proof.

Let b be analytic for σ. The entire function $z \mapsto \varphi(\sigma_z(b))$ has period $(i\beta)$ because (assuming $1 \in A$ to simplify) the KMS condition implies

$$\varphi(\sigma_z(b)1) = \varphi(1\sigma_{z+i\beta}(b)).$$

Since $\|\phi(\sigma_t(b))\| \leqslant \|b\|$ for $t \in \mathbb{R}$ the function is bounded on the boundary of the strip $0 < \Im z < \beta$ and has period $i\beta$ so it is bounded on $\mathbb{C}$, hence it is constant. $\square$

Warning: the converse is not true.

For $\beta = 0$ the KMS condition is simply a tracial condition that does not involve σ, but it is standard for this case to *require σ-invariance explicitly*, so (KMS$_0$ state) $\equiv$ (σ-invariant trace).

The set of KMS$_\beta$-states

Let Σ_β be the set of KMS$_\beta$-states of (A, σ). Then

- if $\varphi_i \in \Sigma_{\beta_i}$ and $\beta_i \to \beta$ then any weak* limit point of $\{\varphi_i\}_i$ is a KMS$_\beta$-state;
- if A is unital and separable then Σ_β is a Choquet simplex in the state space of A, that is, it is a weak*-closed convex subset and every point in Σ_β is the barycenter of a unique probability measure concentrated on the extremal points of Σ_β;
- a point $\varphi \in \Sigma_\beta$ is extremal if and only if $\pi_\varphi(A)''$ is a factor
- extremal KMS$_\beta$ states are also called pure phases.

Phase transition and symmetry breaking

Phase transition is an abrupt change in physical properties of the system.

Example: transition between the solid, liquid, and gaseous phases as temperature increases.

Phase transitions often (but not always) take place between phases with different symmetry. Some intuitive examples are:

- A snowflake is less symmetric than a (spherical) drop of water.
- Ferromagnets are capable of spontaneous magnetization (dipoles "align" each other) at low temperatures.

In C*-algebraic terms, the group of automorphisms of A commuting with σ and preserving a KMS_β-state changes as β varies.
Typically the symmetry group gets smaller as temperature decreases. This is known as **spontaneous symmetry breaking**.

Example (an infinite system based on the Toeplitz algebra)

- Let $\mathcal{T}$:= C*-algebra of the unilateral shift $S : \xi_n \mapsto \xi_{n+1}$ on $\ell^2(\mathbb{N})$
  also $\cong$ the universal C*-algebra generated by an isometry S (i.e. such that $S^*S = 1$).
- Consider the (periodic) dynamics σ defined via the gauge action of $\mathbb{T}$ on $\mathcal{T}$. It is determined by what it does to S: $\sigma_t(S) = e^{it}S$.

- Notice (by Wick ordering products on S and S^*) that the set $\{S^m S^{*n} : m, n \in \mathbb{N}\}$ spans a dense *-subalgebra.

- $\sigma_z(S^m S^{*n}) = e^{i(m-n)z} S^m S^{*n}$ so the spanning elements are analytic.
- the KMS$_\beta$ condition is $\varphi(S^m S^{*n}) = e^{-m\beta} \varphi(S^{*n} S^m)$

- Using it twice gives $\varphi(S^m S^{*n}) = e^{-(m-n)\beta} \varphi(S^m S^{*n})$

  so φ is a KMS$_\beta$-state iff $\begin{cases} \varphi(S^m S^{*n}) = 0 & \text{for } m \neq n \\ \varphi(S^n S^{*n}) = e^{-n\beta} & \text{for } m = n. \end{cases}$

- A (unique) KMS$_\beta$ state does exist for eachβ.

Example I: The Hecke algebra of Bost and Connes

Consider the inclusion of orientation preserving affine groups :

$$P_{\mathbb{Z}}^{+} := \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \subset \begin{pmatrix} 1 & \mathbb{Q} \\ 0 & \mathbb{Q}_{+}^{*} \end{pmatrix} =: P_{\mathbb{Q}}^{+}$$

This is a **Hecke pair** (each double coset contains finitely many right cosets and left cosets) i.e.

$L(\gamma) := |(P_{\mathbb{Z}}^{+}\gamma P_{\mathbb{Z}}^{+})/P_{\mathbb{Z}}^{+}| = R(\gamma^{-1})$ is finite.

Definition

The *Hecke algebra* $\mathcal{H}_{\mathbb{Q}}$ is the $*$-algebra generated by the linear span of the characteristic functions of double cosets $[\gamma] \in P_{\mathbb{Z}}^{+}\backslash P_{\mathbb{Q}}^{+}/P_{\mathbb{Z}}^{+}$ with

- **convolution:** $(f * g)(\gamma) := \sum_{\gamma_1 \in P_{\mathbb{Z}}^{+}\backslash P_{\mathbb{Q}}^{+}} f(\gamma\gamma_1^{-1})g(\gamma_1)$ where the sum is over right-cosets γ_1;
- **involution:** $f^*(\gamma) := \overline{f(\gamma^{-1})}$;
- **identity:** $1 = [P_{\mathbb{Z}}^{+}]$.

Example I: The Bost-Connes Hecke C*-algebra.

Definition

The **Hecke C*-algebra** $\mathcal{C}_{\mathbb{Q}}$ is the C^*-algebra obtained by completion of $\mathcal{H}_{\mathbb{Q}}$ through the representation on $\ell^2(P^+_{\mathbb{Z}} \backslash P^+_{\mathbb{Q}})$ defined via

$$L_f(\xi)(\gamma) \;=\; (f * \xi)(\gamma) = \sum_{\gamma_1 \in P^+_{\mathbb{Z}} \backslash P^+_{\mathbb{Q}}} f(\gamma\gamma_1^{-1})\xi(\gamma_1)$$

Definition

The **Bost-Connes quantum dynamical system** $(\mathcal{C}_{\mathbb{Q}}, \sigma)$ consists of the C*-algebra $\mathcal{C}_{\mathbb{Q}}$ with the dynamics σ, characterized on double cosets by

$$\sigma_t([\gamma]) = \left(\frac{R(\gamma)}{L(\gamma)}\right)^{it} [\gamma] \qquad \text{for } t \in \mathbb{R}.$$

Example I: The Bost-Connes phase transition

Theorem

1. *For each* $0 < \beta \leqslant 1$ *there is a unique* KMS_β *state of* $(\mathcal{C}_{\mathbb{Q}}, \sigma)$. *It is an injective type* III_1 *factor state, invariant under the action of* $\mathrm{Aut}\,\mathbb{Q}/\mathbb{Z}$.
2. *For each* $1 < \beta \leqslant \infty$ *the extremal* KMS_β *states* $\phi_{\beta,\chi}$ *are parametrized by the complex embeddings* $\chi : \mathbb{Q}^{cycl} \to \mathbb{C}$. *of the maximal cyclotomic extension of* $\mathbb{Q}$. *These are type I factor states, on which the action of* $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \cong \mathrm{Aut}\,\mathbb{Q}/\mathbb{Z}$ *is free and transitive.*
3. *The partition function of the system is the Riemann zeta function.*

The BC system has a phase transition with spontaneous symmetry breaking of a $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ Moreover, the extremal KMS_∞ themselves give the embeddings of $\mathbb{Q}^{ab}$ in $\mathbb{C}$ when evaluated on an "arithmetic $\mathbb{Q}$-subalgebra" of $\mathcal{C}_{\mathbb{Q}}$.

Example II: phase transition of $(C_r^*(\mathcal{O}_K \rtimes \mathcal{O}_K^\times), \sigma^N)$

Affine semigroup: $\mathcal{O}_K \rtimes \mathcal{O}_K^\times$ a.k.a. "$ax + b$" semigroup of $\mathcal{O}_K$ (algebraic integers in K).

Toeplitz-type C*-algebra: $C_r^*(\mathcal{O}_K \rtimes \mathcal{O}_K^\times)$ generated by isometries:

$T_{(b,a)}\xi_{(y,x)} = \xi_{(b+ay,ax)}$ on the Hilbert space $\ell^2(\mathcal{O}_K \rtimes \mathcal{O}_K^\times)$.

Time evolution $\sigma_t(T_{(b,a)}) = [\mathcal{O}_K : a\mathcal{O}_K]^{it} T_{(b,a)}$ for $t \in \mathbb{R}$.

Theorem (Cuntz-Deninger-L, Math. Ann. (2013))

For $\beta > 2$ the KMS$_\beta$ states of $(C_r^*(\mathcal{O}_K \rtimes \mathcal{O}_K^\times), \sigma)$ are affinely isomorphic to the tracial states of

$$\mathcal{A} := \bigoplus_{\gamma \in \mathcal{C}\ell_K} C^*(J_\gamma \rtimes U_K)$$

with $J_\gamma \in \gamma$ an integral ideal representing the ideal class $\gamma \in \mathcal{C}\ell_K$ (we let $\mathcal{O}_K$ represent the trivial class).

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$\mathbb{Z}^n$ actions by linear toral automorphisms

The problem of finding the extremal traces of $C^*(J_\gamma \rtimes U_K)$ contains as a crucial step finding the ergodic invariant measures for a group of linear automorphisms of the d-torus. The acting matrices $\rho(u) \in \mathrm{GL}_d(\mathbb{Z})$ come from embedding units into $\mathbb{C}$, and the d-torus $\mathbb{T}^d$ is the dual of $\mathbb{Z}^d \cong J_\gamma$.
Multiplication by $\rho(u) \in \mathrm{GL}_d(\mathbb{Z})$ does not increase denominators, so rational points in $\mathbb{R}^d$ have finite $\mathbb{Z}^n$-orbits $\pmod{\mathbb{Z}^d}$, and –not so obviously– the converse also holds. (cf. cat maps)

$\mathbb{Z}^n$-actions by toral automorphisms of $\mathbb{T}^d$ that contain a partially hyperbolic element have some obvious ergodic invariant probability measures:

- equidistributions on finite orbits
- Haar measure

Are these all?

Still open: cf. Furstenberg's $\times_2$ $\times_3$ analogous question about ergodic invariant measures on $\mathbb{T}$ for the transformations $\times_2 : z \mapsto z^2$ and $\times_3 : z \mapsto z^3$

Assuming positive entropy, Haar measure is the only e.i.m. :

$\times_2$ $\times_3$ [Rudolph, ETDS '90] , [Johnson, IJM '92]

Higher-dimensional version [Einsiedler-Lindenstrauss, ERA-AMS '03]

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F? = Generalized Furstenberg question for number fields

[L-Warren, JOT 2019]

1. rank $U_K = 0$, ($\mathbb{Q}$ or quadratic imaginary) then $U_K = W$:

   $$\{\text{ergodic invariant measures}\} \longleftrightarrow \hat{\mathcal{O}}_K/W$$ (no F) (boring)

2. rank $U_K = 1$, (real quadr., mixed cubic, complex quartic):

   $U_K \circlearrowright \hat{\mathcal{O}}_K \leftrightsquigarrow$ Bernoulli [Katznelson] (F? = no) (hopeless)

3. CM fields of degree > 4:

   $U_K \circlearrowright \hat{\mathcal{O}}_K$ proper invariant subtori (F? = no, but...) (intriguing)

   reducible case; not much is known in general, [Katok-Spatzier, EDTS (1998)]: (under extra assumptions) extensions of a zero-entropy measure on a torus of smaller dimension with Haar conditional measures on the fibers.

4. $K \neq CM$, rank $U_K \geqslant 2$ (ID fields) (F?) (hopeful)