Breakthrough methods for Noncommutative Calculus

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What is Noncommutative analysis?

- A mathematical framework for studying problems arising from physics, signal analysis and other areas.
- A generalisation of classical calculus, where familiar algebraic identities like $xy = yx$ fail to hold.
- A source of new ideas for classical calculus, geometry, probability etc.

So how is noncommutative analysis used?

What is noncommutativity?

In arithmetic, multiplication is commutative:

$$3 \times 4 = 4 \times 3 = 12.$$

Order of terms does not matter.
For operations, this is not the case.
Putting on socks then shoes is different to putting on shoes then socks.
Exactly the same principle applies to observables in quantum mechanics.

Quantum mechanical observables

Early on in the history of quantum mechanics, physicists realised that noncommutativity was essential to understand the structure of observables:

"[...] rewriting Heisenberg's form of Bohr's quantum condition, I recognized at once its formal significance. It meant that the two matrix products pq and qp are not identical. "

- physicist Max Born, describing his realisation c.1925 that observables in quantum mechanics do not commute.

Modern quantum mechanics

Over the 20th century physcists and mathematicians such as John von Neumann, Paul Dirac and others reformulated quantum mechanics into a sophisticated mathematical theory. The core ideas are:

- Hilbert spaces
- C^*-algebras and von Neumann algebras
- Noncommutative algebra

My work focuses on applications of these ideas to the rest of mathematics. This is a heavily studied but still active area of research.

A puzzle

Suppose that you have two coins:

(If you toss one of them, it returns either heads or tails.)
Toss both at once and count the number of heads.
What are the possible outcomes??

A puzzle

Answer:
There are two possible outcomes:

$$\left\{ \frac{2+\sqrt{2}}{2}, \frac{2-\sqrt{2}}{2} \right\}.$$

Obviously these are no ordinary coins!
(This is not just a word game, there is legitimate mathematics and careful science here.)

About those coins...

That "coin toss" is actually the outcome of measuring the observables:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and:

$$B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

on a two-state quantum system (a *qubit*).

About those coins...

What we have here are two random variables which, when individually measured, yield the values $\{0, 1\}$, but when their sum is measured yields $\{\frac{2+\sqrt{2}}{2}, \frac{2-\sqrt{2}}{2}\}$.

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The outcome of $\{\frac{2+\sqrt{2}}{2}, \frac{2-\sqrt{2}}{2}\}$ is particular to this choice of A and B, other "quantum coins" can exhibit very different behaviour.

Recent applications of noncommutative mathematics

Besides applications to physics, there are applications of noncommutative analysis in other areas. Our group has made contributions to:

- Perturbation theory, and
- Fractal geometry.

Perturbation theory

How do noncommutating operators behave under small perturbations?
The classical Taylor formula states that when ε is small:

$$f(x+\varepsilon) \sim f(x) + f'(x)\varepsilon + \frac{1}{2}f''(x)\varepsilon^2 + \cdots .$$

But when x and y do not commute this is no longer valid. Instead we have:

$$f(A+B) \sim f(A) + \mathcal{T}^{A,A}_{f^{[1]}}(B) + \frac{1}{2}\mathcal{T}^{A,A,A}_{f^{[2]}}(B) + \cdots .$$

The symbol $\mathcal{T}$ denotes a multiple operator integral: a fundamental tool in noncommutative perturbation theory.

Applications of perturbation theory

With double operator integral theory, we can obtain precise answers to questions in quantum physics such as:

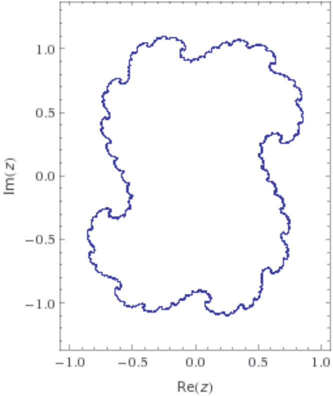
(i) How does the number of quantum states bound by a potential change under small perturbations? (see Birman-Solomyak, J. Soviet Math. (1992)).

(ii) What about when a magnetic field is activated? (see our own recent work, Levitina-S-Zanin arXiv:1703.04254).

Besides this, there are many applications of noncommutative methods outside of mathematical physics.

Fractal geometry

How do you integrate functions on a fractal?



Fractals like the above Julia set are typically *non-rectifiable*, meaning that one cannot measure their length in the usual way.

Fractal geometry

A. Connes proposed a formula for the Hausdorff measure λ_p of a Julia set J:

$$\int_J f(z)d\lambda_p(z) = c_p ⨍ f(Z)|đZ|^p$$

this is a kind of p-dimensional integral, where $1 < p < 2$ is the Hausdorff dimension.

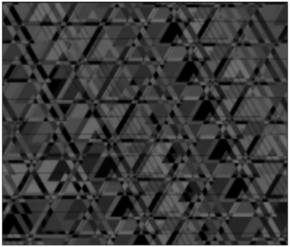
The strange notations $⨍$ and $đ$ come from Connes' quantised calculus, essentially a noncommutative tool.

We have been able to prove this formula for the first time using recently developed noncommutative methods: See Connes-S-Zanin Sb. Math. (2017) and Connes-McDonald-S-Zanin Ergod. Th. & Dynam. Sys. (2017).

Noncommutative geometry

Noncommutative geometry asks the question: how do you study "spaces" when the coordinates do not commute? Examples of such spaces arise in physics, such as the phase space in quantum mechanics.

A fundamental example: "quantum tori"



This picture is supposed to represent a quantum torus somehow, but really these "spaces" cannot be visualised.

Quantum tori

Quantum Tori are the most heavily studied "noncommutative spaces" (i.e., noncommutative algebras with many of the features of algebras of functions on spaces).
Let $0 < \theta < 2\pi$, and consider the two infinite matrices:

$$U = \begin{pmatrix} \ddots & & & & & & \\ & e^{-2i\theta} & & & & & \\ & & e^{-i\theta} & & & & \\ & & & 1 & & & \\ & & & & e^{i\theta} & & \\ & & & & & e^{2i\theta} & \\ & & & & & & \ddots \end{pmatrix}$$

(the blank regions are filled with zeros)

Quantum Tori

and:

$$V = \begin{pmatrix} \ddots & 1 & & & & & \\ & 0 & 1 & & & & \\ & & 0 & 1 & & & \\ & & & 0 & 1 & & \\ & & & & 0 & 1 & \\ & & & & & 0 & \\ & & & & & & \ddots \end{pmatrix}$$

Quantum Tori

The operators U and V satisfy the relation:

$$UV = e^{-2i\theta}VU.$$

Together they generate an algebra called the "noncommutative torus" $C(\mathbb{T}^2_\theta)$.
Despite having little resemblance to a geometric space, there has been great interest in studying $C(\mathbb{T}^2_\theta)$ as a tractable noncommutative space.

Quantum Tori

A number of basic questions concerning calculus on quantum tori have remained open.
For example: Connes proposed a definition of a "quantum differential" $đx$ for $x \in C(\mathbb{T}^2_\theta)$. How do the properties of this $đx$ relate to classical differentiability?

Calculus on quantum tori

One of our most recent results provide an almost complete answer to this question:

Theorem (McDonald-S-Xiong. Com. Math. Phys 2019)

Let $x \in L_2(\mathbb{T}_\theta^d) \cap \dot{W}_d^1(\mathbb{T}_\theta^d)$ be self-adjoint. For any continuous normalized trace φ on $\mathcal{L}_{1,\infty}$ we have

$$\varphi(|\bar{d}x|^d) = c_d \int_{\mathbb{S}^{d-1}} \tau\Big(\big(\sum_{j=1}^d |\partial_j x - s_j \sum_{k=1}^d s_k \partial_k x|^2\big)^{\frac{d}{2}}\Big) ds \approx_d \|x\|_{\dot{W}_d^1}.$$

Comparison to the commutative case

In the commutative ($\theta = 0$) case then the formula for $\varphi(|\bar{d}x|^d)$ has been known since the 1980s (Connes. 1988)

$$\varphi(|\bar{d}f|^d) = c_d \int_{\mathbb{T}^d} \left(\sum_{j=1}^{d} |\partial_j f|^2 \right)^{d/2} dt$$

but the noncommutative case has not been known until recently.

Thank you for listening!