Symmetric Finite Generalised Polygons

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THEOREM (B. KERÉKJÁRTÓ (1941))

Every triply transitive group of continuous transformations of the circle or the sphere is permutationally isomorphic to

- $\mathrm{PGL}(2, \mathbb{R})$ *(circle)*
- $\mathrm{PGL}(2, \mathbb{C})$ *or* $\mathrm{PGL}(2, \mathbb{C}) \rtimes$ (*complex conj.*) *(sphere)*.

Theorem (T. G. Ostrom and A. Wagner (1959))

A finite projective plane admitting a doubly transitive group of automorphisms is Desarguesian.

[1] '2-transitive and flag-transitive designs', Coding theory, design theory, group theory (Burlington, VT), 13–30, Wiley.

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Quote: W. M. Kantor (1993)[1]

"This was the first time 2-transitivity produced a complete classification of finite geometries. Since then the notion of a geometric classification in terms of a group-theoretic hypothesis has become commonplace. That was not the case 35 years ago, and it is a measure of these papers' influence that this type of hypothesis is now regarded as a natural extension of Klein's Erlangen program."

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THEOREM (R. MOUFANG (1932/33); G. PICKERT (1955))

Let Γ be a projective plane and let $G \leqslant \mathrm{Aut}(\Gamma)$. If for every line ℓ, $G_{(\ell)}$ acts transitively on the points of $\Gamma \setminus \ell$, then Γ can be coordinatised by an alternative division ring.

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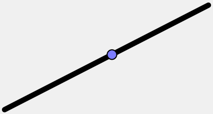
point-wise stabiliser of ℓ

Flag



D. G. Higman & J. E. McLaughlin (1961)

Let Γ be a linear space and $G \leqslant \mathrm{Aut}(\Gamma)$.

G transitive on flags $\implies$ G primitive on points.

Primitive

$G \leqslant \mathrm{Sym}(\Omega)$ does not preserve a partition of Ω, except the trivial ones:

- $\{\Omega\}$
- $\{\{\omega\} : \omega \in \Omega\}$

2-transitive $\implies$ 2-homogeneous $\implies$ primitive $\implies$ quasiprimitive $\implies$ innately transitive $\implies$ semiprimitive $\implies$ transitive

W. M. Kantor (1987)

A projective plane π of order x admitting a point-primitive automorphism group G is Desarguesian and $G \geqslant \mathrm{PSL}(3, x)$, or else G is boring[2].

K. Thas and Zagier 2008

A non-Desarguesian projective plane π with $\mathrm{Aut}(\pi)$ point-primitive has at least 4×10^{22} points.

[2] The number of points $(x^2 + x + 1)$ is a prime and G is a regular or Frobenius group of order dividing $(x^2 + x + 1)(x + 1)$ or $(x^2 + x + 1)x$.

B. Xia (2018)

If there is a finite non-Desarguesian flag-transitive projective plane of order x with $v = x^2 + x + 1$ points, then

- v is prime with $m \equiv 8 \pmod{24}$, and
- there exists a finite field F of characteristic 3, and m elements, satisfying certain polynomial equations.

N. Gill (2016)

If G acts transitively on a finite non-Desarguesian projective plane, then

- the Sylow 2-subgroups of G are cyclic or generalised quaternion, and
- if G is insoluble, then $G/O(G) \cong \mathrm{SL}_2(5), \mathrm{SL}_2(5).2$.

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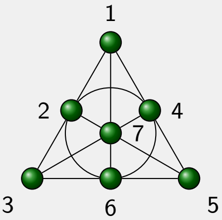
Conjecture; D. Hughes (1959)

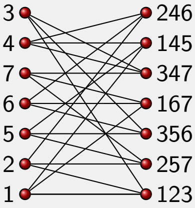
A finite projective plane with a transitive automorphism group is Desarguesian.

SUR LA TRIALITÉ ET CERTAINS GROUPES QUI S'EN DÉDUISENT

Par J. TITS

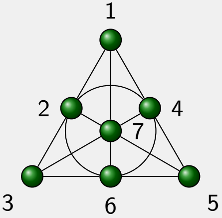
SOMMAIRE

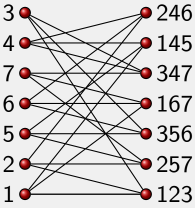




Generalised n-gon:

Incidence graph has girth $= 2 \times$ diameter $= 2n$.





Generalised n-gon:

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Feit-Higman Theorem (1964):

Thick $\implies n \in \{2, 3, 4, 6, 8\}$.

EQUIVALENT DEFINITION

(I) there are no ordinary k-gons for $2 \leqslant k < n$,

(II) any two elements are contained in some ordinary n-gon.

Equivalent definition

(i) there are no ordinary k-gons for $2 \leqslant k < n$,

(ii) any two elements are contained in some ordinary n-gon.

- order (s, t)
  - every line has $s + 1$ points,
  - every point lies on $t + 1$ lines.
- thick if $s, t \geqslant 2$.

- (3) projective planes
- (4) generalised quadrangles
- (6) generalised hexagons
- (8) generalised octagons

Classical examples

3. projective planes

   Desarguesian planes $\to \mathrm{PSL}(3, q)$.

4. generalised quadrangles

   polar spaces associated with $\mathrm{PSp}(4, q)$, $\mathrm{PSU}(4, q)$ *and* $\mathrm{PSU}(5, q)$, *and their duals.*

6. generalised hexagons

   geometries for $G_2(q)$ *and* ${}^3D_4(q)$.

8. generalised octagons

   geometries for ${}^2F_4(q)$.

(3) projective planes
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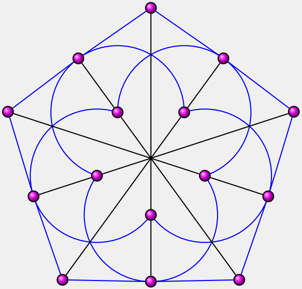
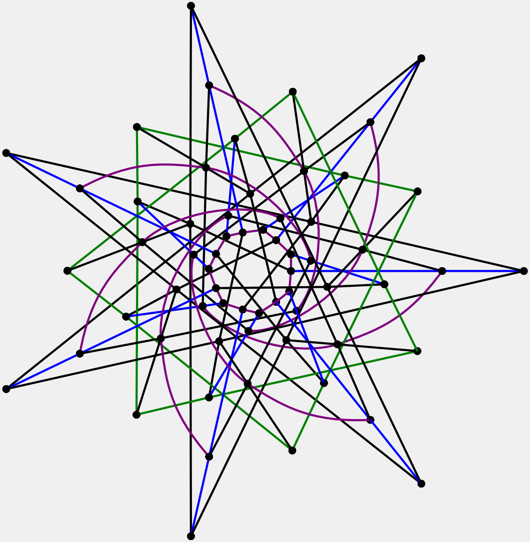
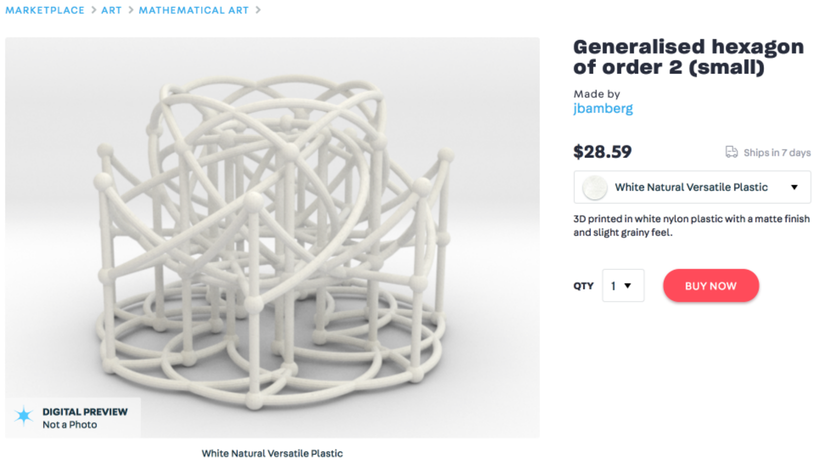
geometries for $G_2(q)$ *and* ${}^3D_4(q)$.

(8) generalised octagons
geometries for ${}^2F_4(q)$.

Many other examples of projective planes and generalised quadrangles known.






MARKETPLACE > ART > MATHEMATICAL ART >
Generalised hexagon of order 2 (small)
Made by
jbamberg
$28.59
Ships in 7 days
White Natural Versatile Plastic
3D printed in white nylon plastic with a matte finish and slight grainy feel.
QTY
1
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Importance

- Building blocks of a building.
- Important to groups of Lie type, in many ways.
- Missing piece of the classification of spherical buildings.
- Many connections to other things in finite geometry and combinatorics.

'CLASSICAL' $\Longrightarrow$

Moufang, flag-transitive, point-primitive, and line-primitive.

MOUFANG FOR GENERALISED QUADRANGLES

For each path (v_0, v_1, v_2, v_3), the group $G_{v_0}^{[1]} \cap G_{v_1}^{[1]} \cap G_{v_2}^{[1]}$ acts transitively on $\Gamma(v_3) \setminus \{v_2\}$.

$G_{v_i}^{[1]}$ is the kernel of the action of G_{v_i} on $\Gamma(v_i)$.

Moufang

↓

Antiflag-transitive

↓

2-transitive

↓

Flag-transitive

↓

Point-primitive

↓

Point-transitive

Projective planes

Moufang

↓

Antiflag-transitive

↓

2-transitive

↓

Flag-transitive

↓

Point-primitive

↓

Point-transitive

Classical (Ostrom-Wagner 1959)

Projective planes

Moufang

↓

Antiflag-transitive

↓

2-transitive — Classical (Ostrom-Wagner 1959)

↓

Flag-transitive — Classical or small group (Kantor 1987)

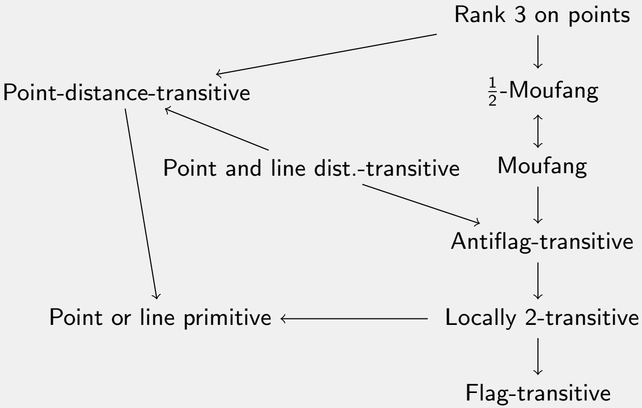
↓

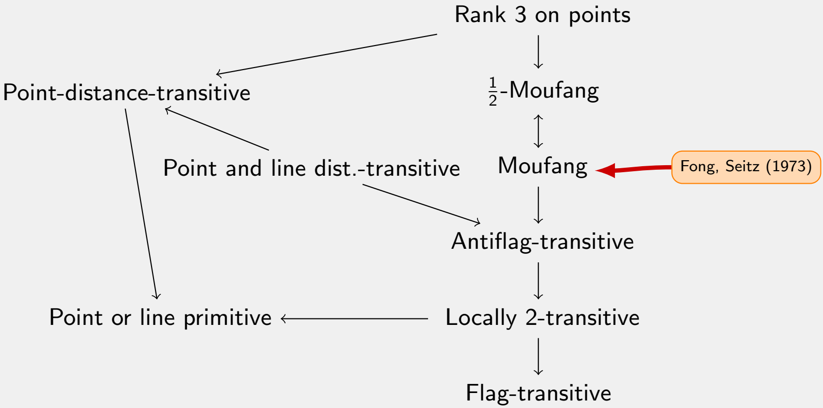
Point-primitive — Classical or small group (Kantor 1987)

↓

Point-transitive

Moufang
↓
Antiflag-transitive
↓
2-transitive — Classical (Ostrom-Wagner 1959)
↓
Flag-transitive — Classical or small group (Kantor 1987)
↓
Point-primitive — Classical or small group (Kantor 1987)
↓
Point-transitive — Classical or some control over minimal normal subgroups (Gill 2007,2016)


Rank 3 on points
Point-distance-transitive
$\frac{1}{2}$-Moufang
Point and line dist.-transitive
Moufang
Antiflag-transitive
Point or line primitive
Locally 2-transitive
Flag-transitive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Fong, Seitz (1973)
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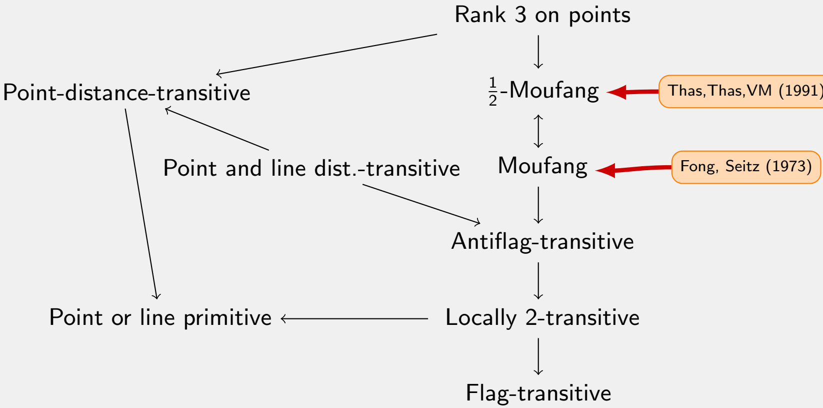
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Thas,Thas,VM (1991)
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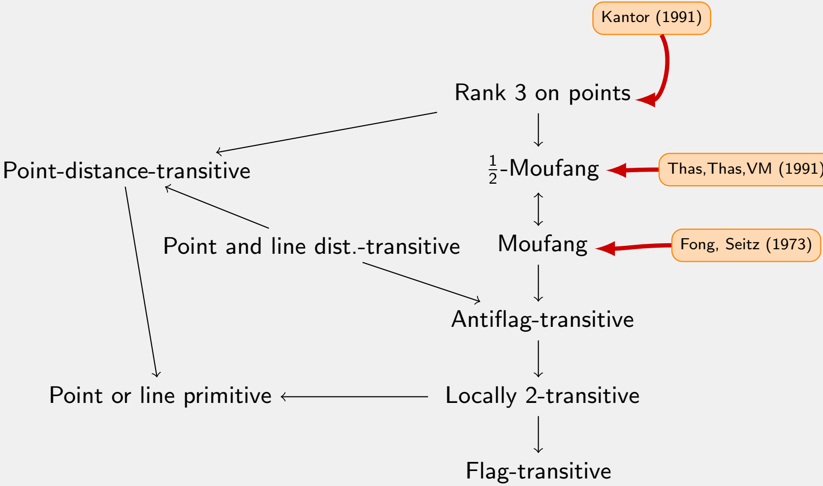
Generalised quadrangles



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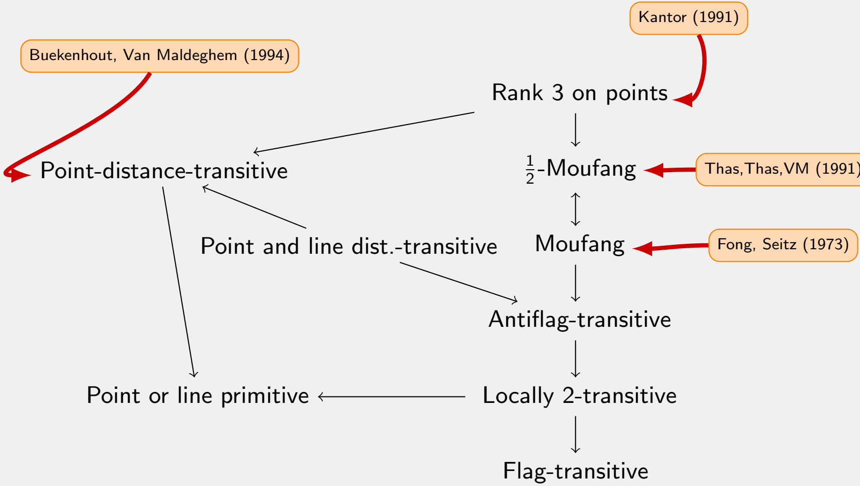

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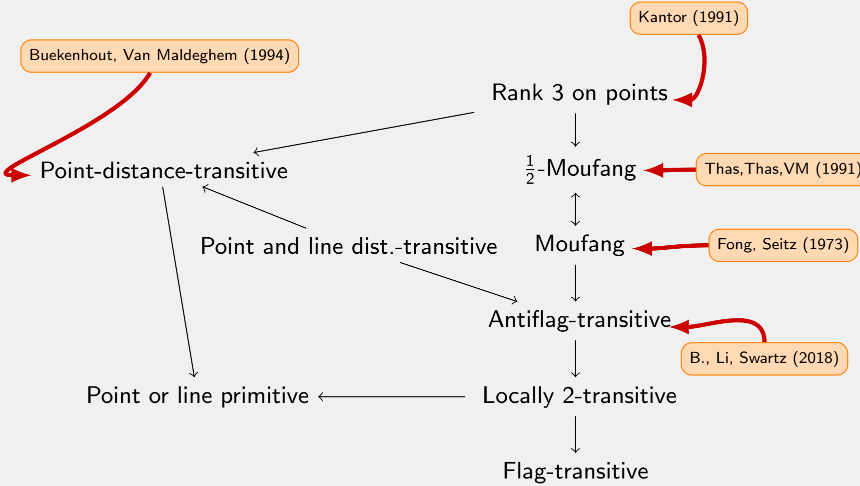

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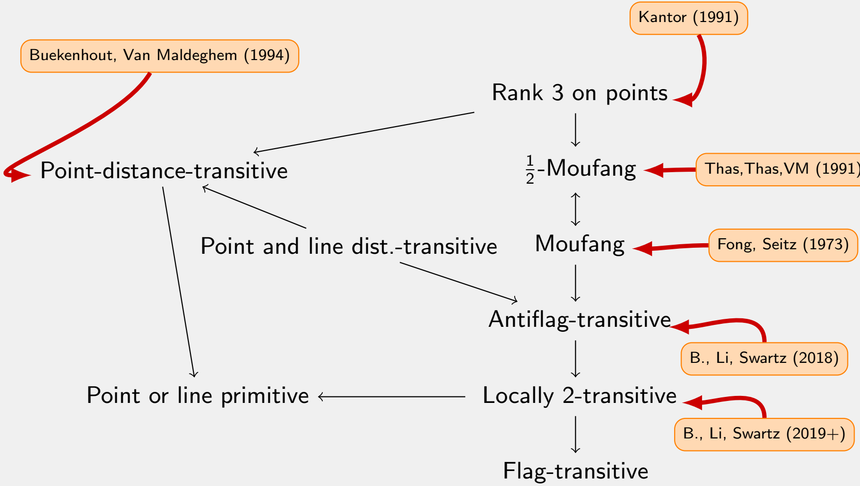
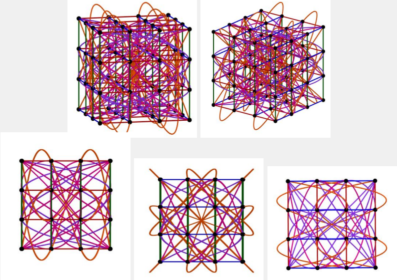


The generalised quadrangle of order $(3, 5)$

- Derived from $Sp(4, 4)$-GQ.
- Automorphism group: $2^6 : (3.A_6.2)$.
- Point-primitive
- Flag-transitive
- Line-imprimitive



Picture courtesy of James Evans.

Fong and Seitz (1973)

A finite thick generalised polygon satisfying the Moufang condition is classical or dual classical.

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Buekenhout-Van Maldeghem (1994)

- A finite thick generalised polygon with a group acting distance-transitively on points is classical or $\mathsf{GQ}(3,5)$.

General results

Fong and Seitz (1973)

A finite thick generalised polygon satisfying the Moufang condition is classical or dual classical.

Buekenhout-Van Maldeghem (1994)

- A finite thick generalised polygon with a group acting distance-transitively on points is classical or $\mathsf{GQ}(3,5)$.
- Distance-transitive $\implies$ point-primitive.

B., Giudici, Morris, Royle, Spiga (2012)

If G acts primitively on the points and lines of a thick GQ then:

- G is almost simple[3].
- If G is also flag-transitive, then G is of Lie type.

[3] G has a unique minimal normal subgroup N, and N is a nonabelian simple group: $N \trianglelefteq G \leqslant \mathrm{Aut}(N)$

For generalised quadrangles

B., Giudici, Morris, Royle, Spiga (2012)

If G acts primitively on the points and lines of a thick GQ then:

- G is almost simple[3].
- If G is also flag-transitive, then G is of Lie type.

Two known flag-transitive GQ's that are point-primitive but line-imprimitive:

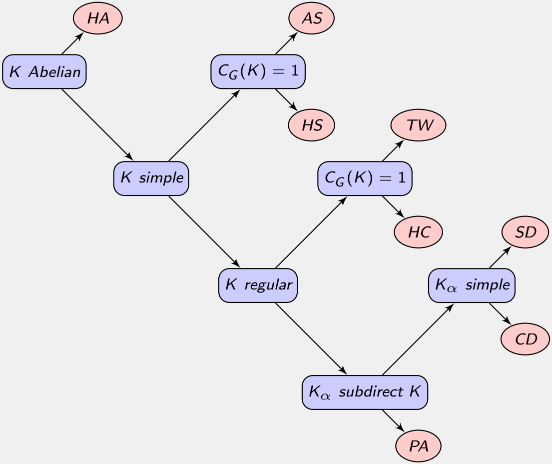
- GQ(3,5),
- GQ of order (15,17) arising from Lunelli-Sce hyperoval.

---

[3] G has a unique minimal normal subgroup N, and N is a nonabelian simple group: $N \trianglelefteq G \leqslant \mathrm{Aut}(N)$

Theorem (The 'O'Nan-Scott' Theorem)

Suppose a finite permutation group G acts primitively on a set Ω. Then one of the following occurs:



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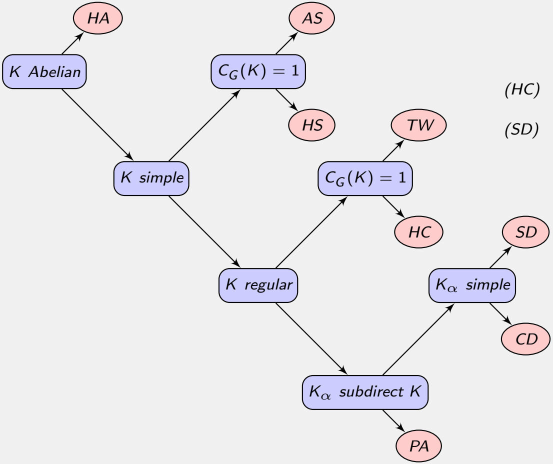
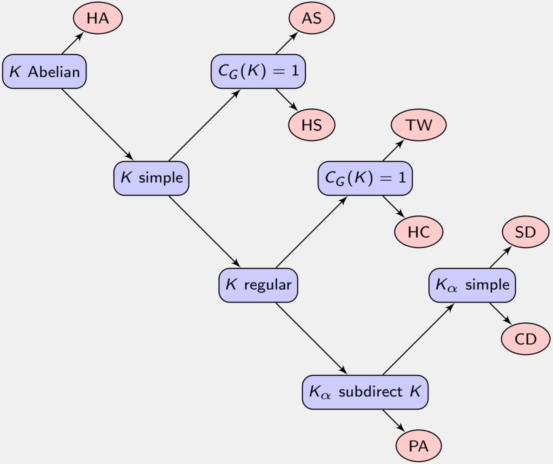
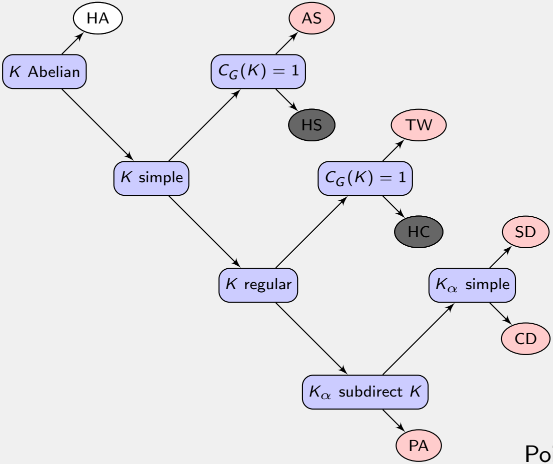


(HC) $K.\mathrm{Inn}(K) \trianglelefteq G \leqslant K.\mathrm{Aut}(K)$
$\Omega = K$, *holomorph action*

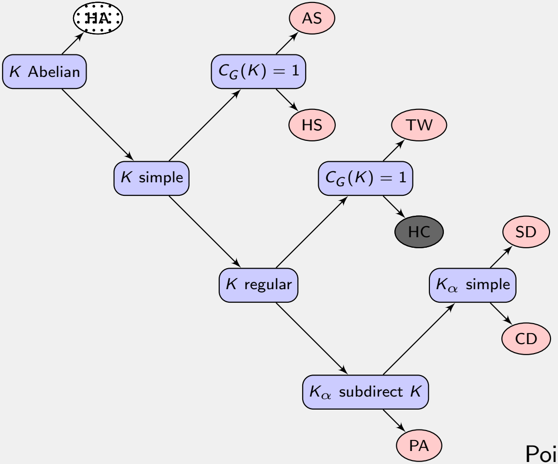
(SD) $K = T^k \trianglelefteq G \leqslant T^k.(\mathrm{Out}(T) \times S_k)$,
$\Omega = T^{k-1}$, *diagonal action*


HA
AS
K Abelian
$C_G(K) = 1$
HS
TW
K simple
$C_G(K) = 1$
HC
SD
K regular
K_α simple
CD
K_α subdirect K
PA



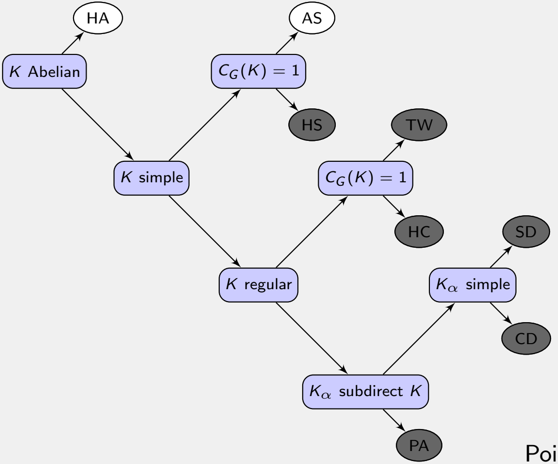
Point-primitive & line-transitive[4]

[4] B., Glasby, Popiel, Praeger 2017



Point-primitive[4]

[4] B., Popiel, Praeger 2019



Point-primitive (aim)

B., Popiel, Praeger (2019)

If G acts primitively on the points of a thick GQ, not affine, then one of the following occurs:

type	soc(G)	necessary conditions
HS	$T \times T$	T has Lie type with Lie rank $\leqslant 7$
SD	T^k	T has Lie type with Lie rank $\leqslant 8$, or $T = \mathrm{Alt}_m$ with $m \leqslant 18$, or T sporadic
CD	$(T^k)^r$	$r \leqslant 3$; T has Lie type with Lie rank $\leqslant 3$, or $T = \mathrm{Alt}_m$ with $m \leqslant 9$, or T sporadic
PA	T^r	$r \leqslant 4$;
AS, TW	–	some information on fixities

Remark

With some extra work, we think HS can be removed completely.

Schneider & Van Maldeghem (2008)

A group acting flag-transitively, point-primitively and line-primitively on a generalised hexagon or octagon is almost simple of Lie type.

Generalised hexagons and octagons

Schneider & Van Maldeghem (2008)
A group acting flag-transitively, point-primitively and line-primitively on a generalised hexagon or octagon is almost simple of Lie type.

B., Glasby, Popiel, Praeger, Schneider (2017)
A group acting point-primitively on a generalised hexagon or octagon is almost simple of Lie type.

SCHNEIDER & VAN MALDEGHEM (2008)
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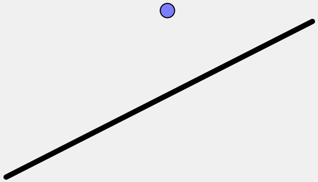
B., GLASBY, POPIEL, PRAEGER, SCHNEIDER (2017)
A group acting point-primitively on a generalised hexagon or octagon is almost simple of Lie type.

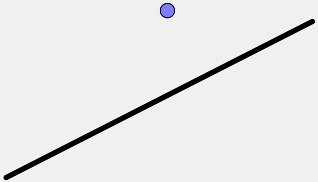
MORGAN & POPIEL (2016)
Moreover, if $T \leqslant G \leqslant \mathrm{Aut}(T)$ with T simple, then

(I) $T \neq {}^2\mathrm{B}_2(q)$ or ${}^2\mathrm{G}_2(q)$;
(II) if $T = {}^2\mathrm{F}_4(q)$, then Γ is the classical generalised octagon or its dual.

Antiflag

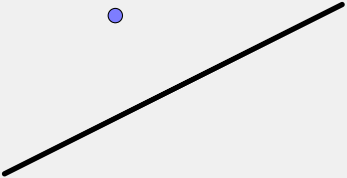


Antiflag

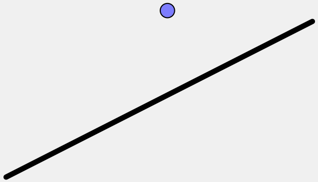


Generalised quadrangle

Given an antiflag (P, ℓ), there is a unique line m on P concurrent with ℓ.



Antiflag



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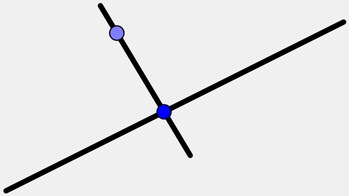

Theorem (B., Li, Swartz 2018)

Let $\mathcal{Q}$ be a finite thick generalised quadrangle and suppose $G \leqslant \mathrm{Aut}(\mathcal{Q})$ acting transitively on the antiflags. Then $\mathcal{Q}$ is classical or $GQ(3,5)$, $GQ(5,3)$.

STRATEGY

- G acts quasiprimitively on points OR lines.
- G point-primitive & line-imprimitive $\implies \mathcal{Q} \cong GQ(3,5)$.
- Reduce to G acting primitively on both points and lines of almost simple type.
- $|T_P|^3 > |T|$ where $\mathrm{soc}(G) = T$; use the characterisation result by Alavi and Burness to determine possibilities for G and G_P.

Groups-on-graphs	Geometry
Locally 3-arc transitive	Antiflag transitive
Locally 2-arc transitive	Transitive on collinear point-pairs and concurrent line-pairs
Edge transitive	Flag transitive

Theorem (B., Li, Swartz (submitted))

If $\mathcal{Q}$ is a thick locally $(G, 2)$-transitive generalised quadrangle, then one of the following holds:

- $\mathcal{Q} \cong GQ(3,5), GQ(5,3)$, *or*
- $\mathcal{Q}$ *is classical.*

STRATEGY

- G acts quasiprimitively on points OR lines.
- G point-quasiprimitive & line-nonquasiprimitive $\implies \mathcal{Q} \cong GQ(3,5)$.
- Reduce to G acting primitively on points, almost simple type, socle of Lie type.
- $|T_P|^3 > |T|$ where $\mathrm{soc}(G) = T$; use the characterisation result by Alavi and Burness to determine possibilities for G and G_P.

1. Show that if G acts flag-transitive on a finite GQ, then G acts primitively on points OR lines.
2. Are all point-primitive GQ's point-distance-transitive?
3. Find new generalised hexagons and octagons.
4. Show that if G acts primitively on the points of a finite GQ, and intransitively on the lines, then the G-orbits on lines divide them in half. (James Evans)