Notes on Tiling Iterated Function Systems

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Abstract. These notes are background for a lecture at the University of Newcastle, November 2019. They relate to a new and unified theory of self-similar tilings and iterated function systems.

1. Introduction

These notes are taken from a paper in preparation on the same topic. They serve for now as background for a new theory of fractal tilings. This theory uses graph iterated function systems (graph IFS) and centers on underlying symbolic shift spaces, in a way that hopefully will be made clear in the Newcastle lecture.

The approach provides a zero dimensional representation of the intricate relationship between shift dynamics on fractals and renormalization dynamics on spaces of tilings. The ideas I will describe unify, simplify, and substantially extend key concepts in foundational papers by Solomyak, Anderson and Putnam, and others. In effect, IFS theory and self-similar tiling theory are unified. These notes and latter ideas are largely new and have not been submitted for publication.

These notes will be supported by illustrations in preparation.

See [**26**] for formal background on iterated function systems (IFS). We are concerned with graph IFS as defined here, but see also [**5, 9, 19, 23, 22, 28, 40**].

2. Foundations

2.1. Graph iterated function systems. Let $\mathcal{F}$ be a finite set of invertible contraction mappings $f:\mathbb{R}^M\rightarrow\mathbb{R}^M$ each with contraction factor $0<\lambda<1$, that is $\|f(x)-f(y)\|\leq\lambda\|x-y\|$ for all $x,y\in\mathbb{R}^M$. We suppose

$$\mathcal{F}=\{f_1,f_2,...,f_N\},\ N>1$$

Let $\mathcal{G}=(\mathcal{E},\mathcal{V})$ be a strongly connected aperiodic directed graph with edges $\mathcal{E}$ and vertices $\mathcal{V}$ with

$$\mathcal{E}=\{e_1,e_2,...,e_N\},\ \mathcal{V}=\{\upsilon_1,\upsilon_2,...,\upsilon_V\},\ 1\leq V<N$$

$\mathcal{G}$ is strongly connected means there is a path, a sequence of consecutive directed edges, from any vertex to any vertex. $\mathcal{G}$ is aperiodic means that if $\mathcal{W}$ is the $V\times V$ matrix whose ij^{th} entry is the number of edges directed from vertex j to vertex i, then there is some power of $\mathcal{W}$ whose entries are all strictly positive.

We call $(\mathcal{F},\mathcal{G})$ a *graph IFS*. The directed graph $\mathcal{G}$ provides the orders in which functions of $\mathcal{F}$ may be composed. The sequence of successive directed edges $e_{\sigma_1}e_{\sigma_2}\cdots e_{\sigma_k}$ is associated with the composite function

$$f_{\sigma_1}f_{\sigma_2}\cdots f_{\sigma_k}:=f_{\sigma_1}\circ f_{\sigma_2}\circ\cdots\circ f_{\sigma_k}$$

The edges may be referred to by their indices $\{1,2,...,N\}$ and the vertices by $\{1,2,...,V\}$.

2.2. Paths in $\mathcal{G}$. Let $\mathbb{N}$ be the strictly positive integers and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. For $N\in\mathbb{N}$, $[N]:=\{1,2,\dots,N\}$.

Σ is the set of directed paths in $\mathcal{G}$, each with an initial vertex. A path $\sigma\in\Sigma$ is written $\sigma=\sigma_1\sigma_2\cdots$ corresponding to the sequence of successive directed edges $e_{\sigma_1}e_{\sigma_2}\cdots$ in $\mathcal{G}$. The length of σ is $|\sigma|\in\mathbb{N}_0\cup\{\infty\}$. A metric d_Σ on Σ is

$$d_\Sigma(\sigma,\omega):=2^{-\min\{k\in\mathbb{N}:\tilde{\sigma}_k\neq\tilde{\omega}_k\}}\text{for } \sigma\neq\omega,$$

where $\tilde{\sigma}_k=\sigma_k$ for all $k\leq|\sigma|$, $\tilde{\sigma}_k=0$ for all $k>|\sigma|$. Then (Σ,d_Σ) is a compact metric space.

The set $\Sigma_*\subset\Sigma$ is the directed paths of finite lengths, and $\Sigma_\infty\subset\Sigma$ is the directed paths of infinite length. For $\sigma\in\Sigma$, let $\sigma^-\in\mathcal{V}$ be the initial vertex and, if $\sigma\in\Sigma_*$, let $\sigma^+\in\mathcal{V}$ be the terminal vertex; and for $v\in\mathcal{V}$ let

$$\Sigma_v:=\{\sigma\in\Sigma_\infty:\sigma^-=v\}$$

For $\sigma\in\Sigma$, $k\in\mathbb{N}$,

$$\sigma|k:=\begin{cases}\sigma_1\sigma_2...\sigma_k & \text{if } |\sigma|>k\\ \sigma_1^+ & \text{if } |\sigma|\leq k\end{cases}$$

$$f_{\sigma|k}:=\begin{cases}f_{\sigma_1}f_{\sigma_2}\cdots f_{\sigma_k} & \text{if } |\sigma|>k\\ f_{\sigma_1^+} & \text{if } |\sigma|\leq k\end{cases},\ f_v:=\chi_{A^v}$$

where I is the identity on $\mathbb{R}^M$ and χ_{A_v} is the characteristic function of $A_v\subset\mathbb{R}^M$, see Definition 1(iii).

$\mathcal{G}^\dagger=(\mathcal{E}^\dagger,\mathcal{V})$ is the graph $\mathcal{G}$ modified so that the directions of all edges are reversed. The superscript $\dagger$ means that the superscripted object relates to $\mathcal{G}^\dagger$. For example, $\Sigma_*^\dagger$ is the set of directed paths in $\mathcal{G}^\dagger$ of finite length, $\Sigma_\infty^\dagger$ is the set of directed paths in $\mathcal{G}^\dagger$, each of which starts at a vertex and is of infinite length, and $\Sigma^\dagger=\Sigma_*^\dagger\cup\Sigma_\infty^\dagger$. While $\mathcal{G}$ is associated with compositions of functions in $\mathcal{F}$, in this paper $\mathcal{G}^\dagger$ is associated with compositions of their inverses.

2.3. Addresses and Attractors. Let $\mathbb{H}$ be the nonempty compact subsets of $\mathbb{R}^M$ and let $d_\mathbb{H}$ be the Hausdorff metric. Singletons in $\mathbb{H}$ are identified with points in $\mathbb{R}^M$.

DEFINITION 1. The **attractor** A of the graph IFS $(\mathcal{F},\mathcal{G})$, its **components** A_v, and the **address** map $\pi:\Sigma\cup\mathcal{V}\to\mathbb{H}$, are defined as follows.

$$\text{(i) } \pi(\sigma):=\lim_{k\to\infty}f_{\sigma|k}(x) \text{ for } \sigma\in\Sigma_\infty,\ \text{fixed } x\in\mathbb{R}^M$$

$$\text{(ii) } A:=\pi(\Sigma_\infty)$$

$$\text{(iii) } \pi(v):=A_v:=\pi(\Sigma_v) \text{ for all } v\in\mathcal{V}$$

$$\text{(iv) } \pi(\sigma):=f_\sigma(A_{\sigma^+}) \text{ for all } \sigma\in\Sigma_*$$

THEOREM 1. *Let* $(\mathcal{F},\mathcal{G})$ *be a graph IFS.*
(1) $\pi:\Sigma\cup\mathcal{V}\rightarrow\mathbb{H}$ *is well-defined*
(2) $\pi:\Sigma\cup\mathcal{V}\rightarrow\mathbb{H}$ *is continuous*
(3) $\pi(\sigma)=\bigcap_{k=1}^{|\sigma|}\pi(\sigma|k)$ *for all* $\sigma\in\Sigma$
(4) $f_{\sigma}(A_{\sigma^{+}})\subset A_{\sigma^{-}}$ *for all* $\sigma\in\Sigma_{*}$

PROOF. (1) For all $\sigma\in\Sigma_{\infty}$, $\pi(\sigma)$ is well-defined by (i), independently of x, because $\mathcal{F}$ is strictly contractive, [**26**]. It follows that A is well-defined by (ii). Also it follows that A_v and $\pi(v)$ are well-defined by (iii), for all $v\in\mathcal{V}$. In turn, $\pi(\sigma)$ is well-defined for all $\sigma\in\Sigma_{*}$ by Definition 1(iv).

(2) π is continuous because for all $\sigma\in\Sigma_{\infty}$

$$d_{\mathbb{H}}(\pi(\sigma|k),\pi(\sigma|l))\leq\lambda^{\min\{k,l\}}\max_{v,w}d_{\mathbb{H}}(A_v,A_w)$$

(3) and (4) follow from Definition 1(iv). □

DEFINITION 2. Define $\sigma\in\Sigma_{\infty}$ to be **disjunctive** if, given any $k\in\mathbb{N}$ and $\theta\in\Sigma_k^{\dagger}$, there is $p\in\mathbb{N}$ so that $\theta=\sigma_p\sigma_{p+1}...\sigma_{p+k}$.

Likewise, $\theta\in\Sigma_{\infty}^{\dagger}$ is disjunctive if, given any $k\in\mathbb{N}$ and $\sigma\in\Sigma_k$, there is $p\in\mathbb{N}$ so that $\sigma=\theta_p\theta_{p+1}...\theta_{p+k}$.

THEOREM 2. *Let* $(\mathcal{F},\mathcal{G})$ *be a graph IFS. Let* $\theta\in\Sigma_{\infty}^{\dagger}$, $x_0\in\mathbb{R}^M$, *and* $x_n=f_{\theta_n}(x_{n-1})$ *for all* $n\in\mathbb{N}$. *Then*

$$\bigcap_{k\in\mathbb{N}}\overline{(\bigcup_{n=k}^{\infty}x_n)}\subseteq A$$

with equality when $\theta\in\Sigma_{\infty}^{\dagger}$ *is disjunctive.*

PROOF. $\Omega(\{x_n:n\in\mathbb{N}\}):=\bigcap_{k\in\mathbb{N}}\overline{(\bigcup_{n=k}^{\infty}x_n)}$ is an $\Omega-$limit set. Specifically it is the set of accumulation points of $\{x_n:n\in\mathbb{N}\}$ in $\mathbb{R}^M$. Since π is continuous

$$\begin{aligned}\Omega\left(\{x_n:n\in\mathbb{N}\}\right)&=\Omega\left(\left\{f_{\theta_n\theta_{n-1}\cdots\theta_1}(x_0):n\in\mathbb{N}\right\}\right)\\&=\pi(\Omega\left(\{\theta_n\theta_{n-1}\cdots\theta_1:n\in\mathbb{N}\}\right))\end{aligned}$$

The $\Omega-$limit set of $\{\theta_n\theta_{n-1}\cdots\theta_1:n\in\mathbb{N}\}$ is contained in or equal to Σ_{∞}, with equality when $\theta\in\Sigma_{\infty}^{\dagger}$ is disjunctive. □

The identity in Theorem 2 underlies the Chaos Game algorithm for calculating pictures of attractors, see [**12**].

2.4. Shift maps.

DEFINITION 3. The **shift map** $S:\Sigma\cup\mathcal{V}\rightarrow\Sigma\cup\mathcal{V}$ is defined by $S(\sigma_1\sigma_2\cdots)=\sigma_2\sigma_3\cdots$ for all $\sigma\in\Sigma, Sv=v$ for all $v\in\mathcal{V}$, with the conventions

$$S^k\sigma=\sigma|k=\sigma_1^{+}\text{ when }k\geq|\sigma|$$

THEOREM 3. *Let* $(\mathcal{F},\mathcal{G})$ *be a graph IFS.*
(1) $S:\Sigma\cup\mathcal{V}\rightarrow\Sigma\cup\mathcal{V}$ *is well-defined*
(2) $S(\Sigma\cup\mathcal{V})=\Sigma\cup\mathcal{V}$
(3) $S:\Sigma\cup\mathcal{V}\rightarrow\Sigma\cup\mathcal{V}$ *continuous*
(4) $f_{\sigma|k}\circ\pi\circ S^k(\sigma)=\pi(\sigma)$ *for all* $\sigma\in\Sigma$, *for all* $k\in\mathbb{N}_0$

PROOF. (1) and (2) can be checked.

(3) S is continuous at every point in $\Sigma_* \cup \mathcal{V}$ because this subset of $\Sigma \cup \mathcal{V}$ is discrete and it is mapped onto itself by S. A calculation using the metric d_Σ proves that S is continuous at every point in Σ_∞.

(4) If $\sigma = \sigma_1$ and $k = 0$ then

$$f_{\sigma|k} \circ \pi \circ S^k(\sigma) = \chi_{A_{\sigma_1^+}} \circ \pi\left(\sigma_1^+\right) = \chi_{A_{\sigma_1^+}}(A_{\sigma_1^+}) = \pi(\sigma_1)$$

If $\sigma = \sigma_1$ and $k = 1$, then

$$f_{\sigma|k} \circ \pi \circ S^k(\sigma) = f_{\sigma_1} \circ \pi\left(\sigma_1^+\right) = f_{\sigma_1}(A_{\sigma_1^+}) = \pi(\sigma_1)$$

If $\sigma \in \Sigma_\infty$ and $k \in \mathbb{N}$, then

$$\begin{aligned} f_{\sigma|k} \circ \pi \circ S^k(\sigma) &= f_{\sigma_1\sigma_2\cdots\sigma_k}(\pi(\sigma_{k+1}\sigma_{k+2}\cdots)) \\ &= f_{\sigma_1\sigma_2\cdots\sigma_k}(\lim_{m\to\infty} \pi(\sigma_{k+1}\sigma_{k+2}\cdots\sigma_m)) \\ &= \lim_{m\to\infty} \pi(\sigma_1\sigma_2\cdots\sigma_m) = \pi(\sigma) \end{aligned}$$

The remaining cases follow similarly. □

2.5. Disjunctive orbits, ergodicity, subshifts of finite type. Let $T = S|_{\Sigma_\infty}$. The dynamical system $T : \Sigma_\infty \to \Sigma_\infty$ is chaotic in the purely topological sense of Devaney [**24**]: it has a dense set of periodic points, it is sensitively dependent on initial conditions, and it is topologically transitive. Topologically transitive means that if Q and R are open subsets of Σ_∞, then there is $K \in \mathbb{N}$ so that

$$Q \cap T^K R \neq \varnothing$$

This is true because the set of disjunctive points in Σ_∞ is dense in Σ_∞ and the orbit under T of a disjunctive point passes arbitrarily close to any given point in Σ_∞.

However, $T : \Sigma_\infty \to \Sigma_\infty$ also possesses many invariant normalized Borel measures, each having support Σ_∞ and such that T is ergodic with respect to each. An example of such a measure $\mu_\mathcal{P}$ may be constructed by defining a Markov process on Σ_∞ using $\mathcal{G}$ and probabilities $\mathcal{P} = \{p_e > 0 : e \in \mathcal{E}\}$ where $\sum_{\substack{d^+=e^+ \\ d\in\mathcal{E}}} p_d = 1$ for all $e \in \mathcal{E}$. Then $\mu_\mathcal{P}$ is the unique normalized measure on the Borel subsets $\mathcal{B}$ of Σ_∞ such that

$$\mu_\mathcal{P}(b) = \sum_{e\in\mathcal{E}} p_e \mu_\mathcal{P}(eb \cap \Sigma_\infty) \text{ for all } b \in \mathcal{B}$$

where $eb := \{\sigma \in \Sigma_\infty : \sigma_1 = e, S\sigma \in b\}$. In particular, $\mu_\mathcal{P}$ is invariant under T, that is

$$\mu_\mathcal{P}(b) = \mu_\mathcal{P}(T^{-1}b) \text{ for all } b \in \mathcal{B}$$

The key point (1) in Theorem 4 is well known: T is ergodic with respect to μ. That is, if $Tb = T^{-1}b$ for some $b \in \mathcal{B}$, then either $\mu_\mathcal{P}(b) = 0$ or $\mu_\mathcal{P}(b) = 1$. As a consequence, the set of disjunctive points has full measure, independent of $\mathcal{P}$.

THEOREM 4. *Let $(\mathcal{F}, \mathcal{G})$ be a graph IFS. Let $(\Sigma_\infty, \mathcal{B}, T, \mu_\mathcal{P})$ be the dynamical system described above. Let D be the disjunctive points in Σ_∞. Then*

(1) ***Parry*** [**30**] *$(\Sigma_\infty, \mathcal{B}, T, \mu_\mathcal{P})$ is ergodic*

(2) $D = TD = T^{-1}D \in \mathcal{B}$

(3) $\mu_\mathcal{P}(D) = 1$, and $\mu_\mathcal{P}(\Sigma_\infty \backslash D) = 0$

Proof. (1) This is a standard result in ergodic theory, see for example [**30**].

(2) It is readily checked that $D \in \mathcal{B}$ and that $T^{-1}D = D = TD$.

(3) Since $(\Sigma_\infty, \mathcal{B}, T, \mu)$ is ergodic and $D = T^{-1}D$, it follows that $\mu(D) \in \{0,1\}$. Also we have

$$1 = \mu(\Sigma_\infty) = \mu(D) + \mu(\Sigma_\infty \backslash D)$$

So either $\mu(D) = 1$ and $\mu(\Sigma_\infty \backslash D) = 0$ or vice-versa. Now notice that

$$\Sigma_\infty \backslash D \subset \bigcup_{x \in \Sigma_* \backslash \varnothing} D_x$$

where $D_x = \{\sigma \in \Sigma_\infty : S^n\sigma \notin c[x] \forall n \in \mathbb{N}_0\}$ where $c[x]$ is the cylinder set

$$c[x] := \{z \in \Sigma_\infty : z = xy, y \in \Sigma_\infty\}.$$

In particular

$$\mu(\Sigma_\infty \backslash D) \leq \sum_{x \in \Sigma_*} \mu(D_x)$$

But $\mu(D_x) = 0$ as proved next, so $\mu(\Sigma_\infty \backslash D) = 0$.

Proof that $\mu(D_x) = 0$: Let $f : \Sigma_\infty \to \mathbb{R}$ be defined by $f(\sigma) = 0$ if $\sigma \in c[x]$ and $f(\sigma) = 1$ if $\sigma \in \Sigma_\infty \backslash c[x]$. Since $f \in L_1(\mu)$, by the ergodic theorem we have

$$\int_{\Sigma_\infty} f d\mu = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k\sigma) \text{ for } \mu\text{-almost all } \sigma \in \Sigma_\infty.$$

But $\int f d\mu = 1 - \mu(c[x]) > 0$ because the support of μ is Σ_∞, and Σ_∞ contains a cylinder set disjoint from $c[x]$ because $|\mathcal{E}| \geq 2$, and all cylinder sets have strictly positive measure. Also $f(T^k\sigma) = 0$ for all $x \in D_x$ so

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k\sigma) = 0 \text{ for all } x \in D_x$$

so $\int_{\Sigma_\infty} f d\mu \neq \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k\sigma)$ for all $x \in D_x$, so $\mu(D_x) = 0$. □

3. Tilings

3.1. Similitudes. A *similitude* is an affine transformation $f : \mathbb{R}^M \to \mathbb{R}^M$ of the form $f(x) = \lambda O(x) + q$, where O is an orthogonal transformation and $q \in \mathbb{R}^M$ is the translational part of $f(x)$. The real number $\lambda > 0$, a measure of the expansion or contraction of the similitude, is called its *scaling ratio*. An *isometry* is a similitude of unit scaling ratio and we say that two sets are isometric if they are related by an isometry.

3.2. Tiling iterated function systems.

Definition 4. The graph IFS $(\mathcal{F}, \mathcal{G})$ is said to obey the **open set condition** (OSC) if there are non-empty bounded open sets $\{\mathcal{O}_v : v \in \mathcal{V}\}$ such that for all $d, e \in \mathcal{E}$ we have $f_e(\mathcal{O}_{e^+}) \subset \mathcal{O}_{e^-}$ and $f_e(\mathcal{O}_{e^+}) \cap f_d(\mathcal{O}_{d^+}) = \varnothing$ whenever $e^- = d^-$.

The OSC for graph IFS is discussed in [**10**] and [**22**].

DEFINITION 5. Let $\mathcal{F} = \{\mathbb{R}^M; f_1, f_2, \cdots, f_N\}$, with $N \geq 2$, be an IFS of contractive similitudes where the scaling factor of f_n is $\lambda_n = s^{a_n}$ where $a_n \in \mathbb{N}$ and $\gcd\{a_1, a_2, \cdots, a_N\} = 1$. Let the graph IFS $(\mathcal{F}, \mathcal{G})$ obeys the OSC. Let

$$A_v \cap A_w = \varnothing \tag{3.1}$$

for all $v \neq w$, and let A_v span $\mathbb{R}^M$. Then $(\mathcal{F}, \mathcal{G})$ is called a **tiling iterated function system** (*tiling IFS).*

The requirement $A_v \cap A_w = \varnothing$ whenever $v \neq w$ is without loss of generality in the following sense. By means of changes of coordinates applied to some of the maps of the IFS, we can move A_v to $T_v A^v$, where $T_v : \mathbb{R}^M \to \mathbb{R}^M$ is a translation, while holding A_w fixed for all $w \neq v$. To do this, let

$$\widetilde{f}_e = \begin{cases} T_v f_e T_v^{-1} & \text{if } e^+ = v \text{ and } e^- = v \\ T_v f_e & \text{if } e^+ \neq v \text{ and } e^- = v \\ f_e T_v^{-1} & \text{if } e^+ = v \text{ and } e^- \neq v \\ f_e & \text{if } e^+ \neq v \text{ and } e^- \neq v \end{cases}$$

and let $\widetilde{\mathcal{F}} = \{f_e : e \in \mathcal{E}\}$. Then the components of the attractor of $\left\{\widetilde{\mathcal{F}}, \mathcal{G}\right\}$ are $\widetilde{A}_w = A_w$ for $w \neq v$ and $\widetilde{A}_v = T_v A_v$. By repeating this process for each vertex, we can modify the IFS so that different components of the attractor have empty intersections. Only the relative positions of the components are changed, while their geometries are unaltered, and (3.1) holds. This being the case, the OSC is simply "there are non-empty open sets $\{\mathcal{O}_v : v \in \mathcal{V}\}$ such that $f_e(\mathcal{O}_{e^+}) \cap f_d(\mathcal{O}_{d^+}) = \varnothing$ for all $d, e \in \mathcal{V}$ with $d \neq e$".

DEFINITION 6. The **critical set** of the tiling IFS $(\mathcal{F}, \mathcal{G})$ is

$$\mathcal{C} := \bigcup_{\substack{d \neq e \\ d,e \in \mathcal{E}}} f_d(A_{d^+}) \cap f_e(A_{e^+})$$

The following theorem tells us that the critical set of a tiling IFS is small, both topologically and measure theoretically, compared to the attractor.

THEOREM 5. *Let $(\mathcal{F}, \mathcal{G})$ be a tiling IFS, let $\mathcal{C}$ be the critical set and let D be the disjunctive points in Σ_∞.*

*(1)**Mauldin & Williams** [**28**] and **Bedford** [**18**] The Hausdorff dimension $\mathcal{D}_H(A)$ of the attractor A of $(\mathcal{F}, \mathcal{G})$ is the unique $t \in [0, M]$ such that the spectral radius of the matrix*

$$\mathcal{W}_{w,v}(t) = \sum_{\{e \in \mathcal{E} : e^+ = v, e^- = w\}} s^{t a_e}$$

equals one. Also $0 < \mu_{\mathcal{H}}(A) < \infty$ where $\mu_{\mathcal{H}}$ is, up to a strictly positive constant factor, the Hausdorff measure on A.

(2) $\mathcal{C} \subset \pi(\Sigma_\infty \backslash D)$

(3) $\mathcal{C} \cap A^\circ = \emptyset$ where A° is the interior of A

(4) $\mu_{\mathcal{P}}(\pi^{-1}(\mathcal{C})) = 0$ for all $\mathcal{P}$.

(5) If $\sum_v \mathcal{W}_{w,v}(t) = 1$ then $\mu_{\mathcal{H}} = \mu_{\mathcal{P}} \circ \pi^{-1}$ where $\mu_{\mathcal{P}}$ is the stationary measure on Σ_∞ obtained when $p_e = s^{\mathcal{D}_H(A) a_e}$ in the Markov process described before Theorem 4. In this case

$$\mu_{\mathcal{H}}(\mathcal{C}) = 0$$

PROOF. (1) To apply [**28**] there must be exactly one edge of $\mathcal{G}$ incoming to each vertex, but this can always be contrived, without changing the dimension, (or the geometries of the components of the attractor,) as we describe here. If $v \in \mathcal{V}$ is such that $d = \left| \sum_{\substack{d^+ = v \\ d \in \mathcal{E}}} 1 \right| > 1$, then introduce new vertices $v^{(1)}, v^{(2)}, ..., v^{(d)}$ and new components of the attractor $A_{v^{(1)}} = A_{v^{(2)}} = ... = A_{v^{(d)}} = A_v$, and replace the d incoming edges to v, by one incoming edge to each of the new vertices, and one outgoing map from each of the new components of the attractor. Then translate the coincident attractors so that they have empty intersections and modify the maps accordingly, as above, and introducing additional maps (related to the original ones by changes of coordinates but with the same scaling rations). This ensure that there is exactly one inward pointing edge at each vertex of $\mathcal{G}$. This reduces the present situation to that in [**28**], who makes this assumption. Clearly the dimension of the attractor is unaltered.

We also have $0 < \mu_H(A) < \infty$ by [**28**, Theorem 3]. Note that [**28**, Theorem 3] requires a different separation condition than the OSC, but both [**10**, Theorem 2.1] and [**22**] refer to [**28**, Theorem 3] as though the two conditions are equivalent, and we have assumed that this is true.

(2) This is the generalization to the graph-directed case of the definitions and argument in [**7**, Proposition 2.2]. We need the dynamical boundary ∂A of the attractor A of $(\mathcal{F}, \mathcal{G})$, namely

$$\partial A := \overline{\bigcup_{n=1}^{\infty} \mathcal{F}|_{\mathcal{G}}^{-n}(\mathcal{C}) \cap A}$$

where $\mathcal{F}|_{\mathcal{G}}^{-n}(\mathcal{C}) = \bigcup_{\sigma \in \mathcal{G}} f_{\sigma_1}^{-1} f_{\sigma_2}^{-1} ... f_{\sigma_n}^{-1}(\mathcal{C} \cap A_{\sigma_n^-})$. We present the proof in parts (a) and (b) for the case $\mathcal{V} = 1$. The arguments are assumed to carry over to the tiling IFS case.

(a) The OSC implies, for similitudes, the open set $\mathcal{O} = \bigcup_{v \in \mathcal{V}} \mathcal{O}_v$ can be chosen so that $\mathcal{O} \cap A \neq \emptyset$ [**35**], which implies $A \backslash \partial A \neq \emptyset$ because in this case $\mathcal{O} \cap \partial A = \emptyset$ by [**29**, Theorem 2.3 via (iii) implies (i) implies (ii)].

(b) $A \backslash \partial A \neq \emptyset$ implies $\partial A \cap \pi(D) = \varnothing$ because if $x = \pi(\sigma) \in \mathcal{C}$ with $\sigma \in D$ then $\partial A = A$ as in [**7**, Proposition 2.2] Prop 2.2. It follows that $\mathcal{C} \subset \pi(\Sigma_\infty \backslash D)$.

(3) This is [**7**, Proposition 2.1] carried over to the tiling IFS case, using the non-overlappingness of A, namely $A \backslash \partial A \neq \emptyset$.

(4) We have

$$\begin{aligned} \mu_A(\mathcal{C}) &\leq \mu_A(\pi(\Sigma_\infty \backslash D)) \text{ by (2)} \\ &= \mu_H(\pi^{-1}\pi(\Sigma_\infty \backslash D)) \text{ since } \mu_A = \mu_H \circ \pi^{-1} \\ &= \mu_H(\Sigma_\infty \backslash D) = 0 \text{ by Theorem 4 (3)} \end{aligned}$$

(5) Using the thermodynamic formalism [**18**, **find exact reference or use MAGIC01] and the assumption that $\sum_v \mathcal{W}_{w,v}(t) = 1$, we find that $\mu_H = \mu_{\mathcal{P}} \circ \pi^{-1}$ is, up to a positive multiplicative constant, the Hausdorff measure obtained when

$$p_e = s^{\mathcal{D}_H(A) a_e} / \sum_{d^+ = e^+} s^{\mathcal{D}_H(A) a_d}$$

□

3.3. Tilings in this paper. According to Grunbaum and Sheppard [**25**] a tiling is a partition of $\mathbb{R}^2$ by closed sets. Here we consider tilings of subsets of $\mathbb{R}^M$ such as fractal blow-ups where tiles are components of attractors of IFSs. In this case tiles may have empty interiors and the question of what it means for tiles to be non-overlapping has to be answered. Here we simply say that two tiles t_1 and t_2 that belong to a tiling are non-overlapping if their intersection is small both topologically and measure theoretrically, relative to the tiles themselves. This matches the customary situation: in a tiling of $\mathbb{R}^2$ such as a partition into triangles, tiles have positive two-dimensional Lebesgue measure, intersections of distinct tiles have zero two-dimensional Lebesgue measure, and are subsets of their topological boundaries.

3.4. The tiling map. Define subsets of Σ_* as follows:

$$\Omega_k = \{\sigma \in \Sigma_* : \xi^-(\sigma) \leq k < \xi(\sigma)\},\ \Omega_0 = [N]$$
$$\Omega_k^v = \{\sigma \in \Omega_k : \sigma^- = v\},\ \Omega_0^v = \{\sigma_1 \in [N] : \sigma_1^- = v\}$$

for all $k \in \mathbb{N}$, $v \in \mathcal{V}$. Here $\xi : \Sigma_* \to \mathbb{N}_0$ is defined for all $\sigma \in \Sigma_*$ by

$$\xi(\sigma) = \sum_{k=1}^{|\sigma|} a_{\sigma_k},\ \xi^-(\sigma) = \sum_{k=1}^{|\sigma|-1} a_{\sigma_k},\ \xi(\varnothing) = \xi^-(\varnothing) = 0$$

Tilings are associated with $\theta \in \Sigma^\dagger$. Edges in $\Sigma^\dagger$ are directed oppositely to those in Σ. For all $\theta \in \Sigma^\dagger$, $k \in \mathbb{N}$, $k \leq |\theta|$,

$$\theta|k := \theta_1\theta_2\cdots\theta_k,\ \theta|0 := \theta_1^-$$
$$f_{-\theta|k} := f_{\theta_1}^{-1} f_{\theta_2}^{-1} \cdots f_{\theta_k}^{-1},\ f_{\varnothing}^{-1} := I$$

Definition 7. The **tiling map** Π from $\Sigma^\dagger$ to collections of subsets of $\mathbb{H}(\mathbb{R}^M)$ defined as follows. For $\theta \in \Sigma_*^\dagger$,

$$\Pi(\theta) = f_{-\theta}\pi\left(\Omega_{\xi(\theta)}^{\theta^+}\right),\ \Pi(\theta|0) = \pi\left(\Omega_0^{\theta^-}\right)$$

and for $\theta \in \Sigma_\infty^\dagger$,

$$\Pi(\theta) = \bigcup_{k\in\mathbb{N}} \Pi(\theta|k)$$

For $\sigma \in \Omega_{\xi(\theta)}^{\theta^+}$ and $\theta \in \Sigma^\dagger$, the set $f_{-\theta}\pi(\sigma)$ is called a **tile** and $\Pi(\theta)$ is called a **tiling**. The **support** of the tiling $\Pi(\theta)$ is the union of its tiles, and $\Pi(\theta)$ is said to tile its support.

Theorem 6. *Let $(\mathcal{F}, \mathcal{G})$ be a tiling IFS.*

(1) *For all $\theta \in \Sigma_\infty^\dagger$, if $t_1, t_2 \in \Pi(\theta)$ with $t_1 \neq t_2$, then $t_1 \cap t_2$ is small both topologically and measure theoretically, compared to t_1. That is, $\mu_{\mathcal{P}}(t_1 \cap t_2) = 0$ and $t_1^\circ \cap t_2 = \emptyset$ where t° is the interior of t.*

(2) *For all $\theta \in \Sigma_\infty^\dagger$ the sequence of tilings $\{\Pi(\theta|k)\}_{k=1}^\infty$ obeys*

$$\Pi(\theta|0) \subset \Pi(\theta|1) \subset \Pi(\theta|2) \subset \cdots \tag{3.2}$$

(3) *$\Pi(\theta)$ is a tiling of a subset of $\mathbb{R}^M$ that is bounded when $\theta \in \Sigma_*^\dagger$ and unbounded when $\theta \in \Sigma_\infty^\dagger$.*

(4) *For all* $\theta \in \Sigma_{\infty}^{\dagger}$

$$\Pi(\theta) = \lim_{k\to\infty} f_{-\theta|k}(\{\pi(\sigma) : \sigma \in \Omega_{\xi(\theta|k)}, \sigma^- = \theta^+\}) \tag{3.3}$$

(5) *Any tile* $t \in \Pi(\theta)$ *can be written* $t = s^i U A_v$ *for some isometry* $U \in \mathcal{U}$, $i \in \{1, 2, ..., \max a_e\}$ *and* $v \in \mathcal{V}$.

PROOF. (1) $\Pi(\theta|0)$ is a tiling in the sense described in Section 3.3. $\Pi(\theta|0) = \pi\left(\Omega_0^{\theta^-}\right) = \pi(\{e \in [N] : e^- = \theta^-\}) = \{f_e(A_{e^+}) : e^- = \theta^-\}$ has support A_{e^-} and its tiles are supposed to be $\{f_e(A_{e^+}) : e^- = \theta^-\}$. We need to check (i) that they are components of attractors of tiling IFSs and (ii) that their intersections are relatively small. (i) is true because for each $e \in [N]$, the set $f_e(A_{e^+})$ is a component of the attractor of the tiling IFS $(f_e \mathcal{F} f_e^{-1}, \mathcal{G})$. (ii) This follows from Theorem 5 parts (3) and (4).

Similarly, $\Pi(\theta|k)$ and $\Pi(\theta)$ are tilings as in Section 3.3: the tiles are components of attractors of appropriately shifted versions of the original tiling IFS and their intersections are isometric to subsets of the critical set of the original tiling IFS.

(2) The proof is algebraic, idependent of topology, essentially the same as for the case where A_v has nonempty interior [**17**], and similar to the case where $V = 1$ [**16**]. Briefly,

$$\begin{aligned}
\Pi(\theta|k+1) &= \{f_{\theta_1}^{-1}...f_{\theta_{k+1}}^{-1} f_{\sigma_1}..f_{\sigma_{|\sigma|}}(A_{\sigma_{|\sigma|}^+}) : \xi(\sigma_1..\sigma_{|\sigma|-1}) \le \xi(\theta_1..\theta_{|\sigma|}) < \xi(\sigma_1..\sigma_{|\sigma|})\} \\
&\supset \{f_{\theta_1}^{-1}...f_{\theta_k}^{-1} f_{\sigma_2}..f_{\sigma_{|\sigma|}}(A_{\sigma_{|\sigma|}^+}) : \xi(\sigma_2..\sigma_{|\sigma|-1}) \le \xi(\theta_2..\theta_{|\sigma|}) < \xi(\sigma_2..\sigma_{|\sigma|})\} \\
&= \{f_{\theta_1}^{-1}...f_{\theta_k}^{-1} f_{\sigma_1}..f_{\sigma_{|\sigma|-1}}(A_{\sigma_{|\sigma|-1}^+}) : \xi(\sigma_1..\sigma_{|\sigma|-2}) \le \xi(\theta_1..\theta_{|\sigma|-1}) < \xi(\sigma_1..\sigma_{|\sigma|-1})\} \\
&= \Pi(\theta|k)
\end{aligned}$$

(3) For $\theta \in \Sigma_*^{\dagger}$, $\Pi(\theta) = f_{-\theta}\pi\left(\Omega_{\xi(\theta)}^{\theta^+}\right)$ so the support of $\Pi(\theta)$ is $f_{-\theta}(\bigcup\{\pi(\sigma) : \sigma \in \Omega_{\xi(\theta)}^{\theta^+}) = f_{-\theta}A_{\theta^+}$. Here $f_{-\theta}$ is a similitude of expansion factor $|s|^{-\xi(\theta)}$ which diverges with $|\theta|$, and A_{θ^+} spans $\mathbb{R}^M$.

(4) This follows from (3).

(5) For $t \in \Pi(\theta)$ we have $t = f_{-(\theta|k)}f_\sigma(A_v)$ for some k, θ, σ and v, with $\xi^-(\sigma) \le \xi(\theta|k) < \xi(\sigma)$. Here $f_{-(\theta|k)}f_\sigma = s^{-m}U$ where $m = \xi(\theta|k) - \xi(\sigma)$ is an integer that lies between 1 and $\max a_e$ and $U \in \mathcal{U}$ is an isometry of the form $s^m f_{-(\theta|k)} f_\sigma$ for some m. □

4. Continuity properties of $\Pi : \Sigma^{\dagger} \to \mathbb{T}$.

4.1. A convenient compact tiling space.

Let

$$\mathbb{T} = \left\{\Pi(\theta) : \theta \in \Sigma^{\dagger}\right\}$$

Let $\rho : \mathbb{R}^M \to \mathbb{S}^M$ be the usual M-dimensional stereographic projection to the M-sphere, obtained by positioning $\mathbb{S}^M$ tangent to $\mathbb{R}^M$ at the origin. Let $\mathbb{H}(\mathbb{S}^M)$ be the non-empty closed (w.r.t. the usual topology on $\mathbb{S}^M$) subsets of $\mathbb{S}^M$. Let $d_{\mathbb{H}(\mathbb{S}^M)}$ be the Hausdorff distance with respect to the round metric on $\mathbb{S}^M$, so that $(\mathbb{H}(\mathbb{S}^M), d_{\mathbb{H}(\mathbb{S}^M)})$ is a compact metric space. Let $\mathbb{H}(\mathbb{H}(\mathbb{S}^M))$ be the nonempty compact subsets of $(\mathbb{H}(\mathbb{S}^M), d_{\mathbb{H}(\mathbb{S}^M)})$, and let $d_{\mathbb{H}(\mathbb{H}(\mathbb{S}^M))}$ be the associated Hausdorff metric. Then $(\mathbb{H}(\mathbb{H}(\mathbb{S}^M)), d_{\mathbb{H}(\mathbb{H}(\mathbb{S}^M))})$ is a compact metric space. Finally, define a metric $d_{\mathbb{T}}$ on $\mathbb{T}$ by

$$d_{\mathbb{T}}(T_1, T_2) = d_{\mathbb{H}(\mathbb{H}(\mathbb{S}^M))}(\rho(T_1), \rho(T_2))$$

for all $T_1, T_2 \in \mathbb{T}$.

THEOREM 7. $(\mathbb{T}, d_{\mathbb{T}})$ *is a compact metric space.*

PROOF. Straightfoward and omitted. □

See also for example [**1, 20, 34, 36, 39**] where related metrics and topologies are defined.

4.2. Continuity. The following definition generalizes a related concept for the case where A is a topological disk and $|\mathcal{V}| = 1$, see [**14**]. For $\theta \in \Sigma_\infty^\dagger$ define $I(\theta) \subset \Sigma_\infty$ to be the set of limit points of $\{\theta_{l+m}\theta_{l+m-1}...\theta_{m+1} : l, m \in \mathbb{N}\}$. Define

$$H_v = \cup\{f_\sigma^{-1} f_\omega(A_v) : \pi(\sigma), \pi(\omega) \in A^v, \sigma, \omega \in \Sigma, \sigma_1 \neq \omega_1\}.$$

This is the union of all images of A_v under its neighbor maps and is a generalization of the same definition in the case $V = 1$, [**2, 3, 4**]. Define the *central open sets* to be

$$O_v = \{x \in \mathbb{R}^M : d(x, A_v) < d(x, H_v)\}$$

It appears that "$(\mathcal{F}, \mathcal{G})$ obeys the OSC" if and only if "A_v is not contained in H_v for all $v \in \mathcal{V}$ ", see [**2, 3, 4**].

Call $\theta \in \Sigma_\infty^\dagger$ **reversible** if

$$\Sigma_{rev}^\dagger := I(\theta) \cap \{\sigma \in \Sigma_\infty : \pi(\sigma) \subset \cup_v O_v\} \neq \varnothing.$$

Equivalently, $\theta \in \Sigma_{rev}^\dagger$ if the following holds: there exists $\sigma \in \Sigma_\infty$ with $\pi(\sigma) \in \cup_v O_v$ such that, for all $L, M \in N$ there is $m \geq M$ so that

$$\sigma_1\sigma_2...\sigma_L = \theta_{m+L}\theta_{m+L-1}...\theta_1$$

Equivalently, in terms of the notion of "full" words, see [**14**], $\theta \in \Sigma_{rev}^\dagger$ if there is a nonempty compact set $A' \subset \cup_v O_v$ such that for any positive integer M there exists $n > m \geq M$ so that

$$f_{\theta_n} f_{\theta_{n-1}} ... f_{\theta_{m+1}}(A_{\theta_{m+1}^+}) \subset A'.$$

THEOREM 8. *Let $(\mathcal{F}, \mathcal{G})$ be a tiling IFS. Then*

$$\Pi|_{\Sigma_{rev}^\dagger} : \Sigma_{rev}^\dagger \subset \Sigma_\infty^\dagger \to \mathbb{T}$$

is continuous and

$$\Pi : \Sigma_\infty^\dagger \to \mathbb{T}$$

is upper semi-continuous in this sense: if $\Pi(\theta^{(n)})$ is a sequence of tilings that converges to a tiling $T \in \mathbb{T}$ as $\theta^{(n)}$ converges to $\theta \in \Sigma_\infty^\dagger$, then $\Pi(\theta) \subset T$.

PROOF. Proof of upper semi-continuity: let $\{\theta^{(n)}\}$ be a sequence of points in $\Sigma_\infty^\dagger$ that converges to θ and such that $\lim \Pi(\theta^{(n)}) = T$ with respect to the tiling metric. Let m be given. Then there is l_m so that for all $n \geq l_m$ we have $\theta|m = \theta^{(n)}|m$ and hence $\Pi(\theta|m) = \Pi(\theta^{(n)}|m) \subset \Pi(\theta^{(n)})$. Hence we have $\Pi(\theta|m) \subset \lim_{n\to\infty} \Pi(\theta^{(n)})$ and hence, since this is true for all m, $\Pi(\theta) \subset \lim_{n\to\infty} \Pi(\theta^{(n)})$.

Proof that $\Pi|_{\Sigma^{\dagger}_{rev}} : \Sigma^{\dagger}_{rev} \to \mathbb{T}$ is continuous involves blow-ups of central opens sets. Analogously to the definition of Π, define a mapping Ξ from $\Sigma^{\dagger}$ into to collections of subsets of $\mathbb{H}(\mathbb{R}^M)$ as follows. For $\theta \in \Sigma^{\dagger}_{*}, \theta \neq \varnothing$,

$$\Xi(\theta_1\theta_2...\theta_k) := \{f_{-\theta_1\theta_2...\theta_k} f_{\sigma}(\overline{O_{\sigma^+}}) : \sigma \in \Omega_{\xi(\theta_1\theta_2...\theta_k)}, \theta_k^- = \sigma_1^-\},$$

and for $\theta \in \Sigma^{\dagger}_{\infty}$

$$\Xi(\theta) := \bigcup_{k\in\mathbb{N}} \Xi(\theta|k).$$

As is the case for Π, increasing families of sets are obtained: each collection $\Xi(\theta)$ comprises a covering by compact sets of a subset of $\mathbb{R}^M$, the subset being bounded when $\theta \in \Sigma^{\dagger}_{*}$ and unbounded when $\theta \in \Sigma^{\dagger}_{\infty}$. For all $\theta \in \Sigma^{\dagger}_{\infty}$ the sequence of sets $\{\Xi(\theta|k)\}_{k=1}^{\infty}$ is nested according to

$$\Xi(\theta|1) \subset \Xi(\theta|2) \subset \Xi(\theta|3) \subset \cdots .$$

and we have $\{\Xi(\theta|k)\}$ converges to $\Xi(\theta)$ in the metric introduced in Section 4.1. In particular, when reversible, the new tiles, those in $\Xi(\theta|k+1)$ that are not in $\Xi(\theta|k)$, are located further and further away from the origin as k increases. The result follows. □

5. Symbolic structure : canonical symbolic tilings and symbolic inflation and deflation

Write $\Omega_k^{(v)}$ to mean any of Ω_k^v or Ω_k. The following lemma tells us that $\Omega_{k+1}^{(v)}$ can be obtained from $\Omega_k^{(v)}$ by adding symbols to the right-hand end of some strings in $\Omega_k^{(v)}$ and leaving the other strings unaltered.

Lemma 1. ***(Symbolic Splitting)*** *For all $k \in \mathbb{N}$ and $v \in \mathcal{V}$ the following relations hold:*

$$\Omega_{k+1}^{(v)} = \left\{\sigma \in \Omega_k^{(v)} : k+1 < \xi(\sigma)\right\} \cup \left\{\sigma j \in \Sigma_*^{(v)} : \sigma \in \Omega_k^{(v)}, k+1 = \xi(\sigma)\right\}.$$

Proof. Follows at once from definition of $\Omega_k^{(v)}$. □

Define $\alpha_s^{-1} : \Omega_k^{(v)} \to 2^{\Omega_{k+1}^{(v)}}$ by

$$\alpha_s^{-1}\sigma = \sigma \text{ if } k+1 < \xi(\sigma)$$
$$\alpha_s^{-1}\sigma = \{\sigma e : \sigma_{|\sigma|}^{+} = e^-, e \in \mathcal{E}\} \text{ if } k+1 = \xi(\sigma)$$

Then

$$\alpha_s^{-1}(\Omega_k^v) = \Omega_{k+1}^v$$

This defines symbolic inflation or "splitting and expansion" of $\Omega_k^{(v)}$, some words in $\Omega_{k+1}^{(v)}$ being the same as in $\Omega_k^{(v)}$ while all the others are split. The inverse operation is symbolic deflation or "amalgamation and shrinking", described by a function

$$\alpha_s : \Omega_{k+1}^{(v)} \to \Omega_k^{(v)},\ \alpha_s(\Omega_{k+1}^{(v)}) = \Omega_k^{(v)}$$

where $\alpha_s(\sigma)$ is the unique $\omega \in \Omega_k^{(v)}$ such that $\sigma = \omega\beta$ for some $\beta \in \Sigma_*$. Note that β may be the empty string.

We can use $\Omega_k^{(v)}$ to define a partition of $\Omega_m^{(v)}$ for $m \geq k$. The partition of $\Omega_{k+j}^{(v)}$ is $\Omega_{k+j}^{(v)}/\sim$ where $x \sim y$ if $\alpha_s^j(x) = \alpha_s^j(y)$.

COROLLARY 1. ***(Symbolic Partitions)*** *For all $m \geq k \geq 0$, the set $\Omega_k^{(v)}$ defines a partition $P_{m,k}^{(v)}$ of $\Omega_m^{(v)}$ according to $p \in P_{m,k}^{(v)}$ if and only if there is $\omega \in \Sigma_*$ such that*

$$p = \{\omega\beta \in \Omega_m^{(v)} : \beta \in \Omega_k^{(v)}\}.$$

PROOF. This follows from Lemma 1: for any $\theta \in \Omega_m^{(v)}$ there is a unique $\omega \in \Omega_k^{(v)}$ such that $\theta = \omega\beta$ for some $\beta \in \Sigma_*$. Each word in $\Omega_m^{(v)}$ is associated with a unique word in $\Omega_k^{(v)}$. Each word in $\Omega_k^{(v)}$ is associated with a set of words in $\Omega_m^{(v)}$. □

According to Lemma 1, $\Omega_{k+1}^{(v)}$ may be calculated by tacking words (some of which may be empty) onto the right-hand end of the words in $\Omega_k^{(v)}$. We can invert this description by expressing $\Omega_k^{(v)}$ as a union of predecessors ($\Omega_j^{(v)}$s with $j < k$) of $\Omega_k^{(v)}$ with words tacked onto their other ends, that is, their left-hand ends.

COROLLARY 2. ***(Symbolic Predecessors)*** *For all $k \geq a_{\max} + l$, for all $v \in \mathcal{V}$, for all $l \in \mathbb{N}_0$,*

$$\Omega_k^{(v)} = \bigsqcup_{\omega \in \Omega_l^{(v)}} \omega \Omega_{k-\xi(\omega)}^{\omega^+}$$

PROOF. It is easy to check that the r.h.s. is contained in the l.h.s.

Conversely, if $\sigma \in \Omega_k^{(v)}$ then there is unique $\omega \in \Omega_l^{(v)}$ such that $\sigma = \omega\beta$ for some $\beta \in \Sigma_*$ by Corollary 1. Because $\omega\beta \in \Sigma_*$ it follows that β_1 is an edge that starts where the last edge in ω is directed, namely the vertex ω^+. Finally, since $\xi(\omega\beta) = \xi(\omega) + \xi(\beta)$ it follows that $\beta \in \Omega_{k-\xi(\omega)}^{\omega^+}$. □

6. Canonical tilings and their relationship to $\Pi(\theta)$

DEFINITION 8. We define the **canonical tilings** of the tiling IFS $(\mathcal{F}, \mathcal{G})$ to be

$$T_k = s^{-k}\pi(\Omega_k),\ T_k^v := s^{-k}\pi(\Omega_k^v)$$

$k \in \mathbb{N}, v \in \mathcal{V}$.

A canonical tiling may be written as a disjoint union of isometries applied of other canonical tilings as described in Lemma 2.

LEMMA 2. *For all $k \geq a_{\max} + l$, for all $l \in \mathbb{N}_0$, for all $v \in \mathcal{V}$*

$$T_k^v = \bigsqcup_{\omega \in \Omega_l^v} E_{k,\omega} T_{k-\xi(\omega)}^{\omega^+} \text{ and } T_k = \bigsqcup_{\omega \in \Omega_l} E_{k,\omega} T_{k-\xi(\omega)}^{\omega^+}$$

where $E_{k,\omega} = s^{-k} f_\omega s^{k-\xi(\omega)} \in \mathcal{U}$ is an isometry.

PROOF. Direct calculation using Corollary 2. □

THEOREM 9. *For all $\theta \in \Sigma_*^\dagger$,*

$$\Pi(\theta) = E_\theta T_{\xi(\theta)}^{\theta_{|\theta|}^-},$$

where $E_\theta = f_{-\theta} s^{\xi(\theta)}$. Also if $l \in \mathbb{N}_0$, and $\xi(\theta) \geq a_{\max} + l$, then

$$\Pi(\theta) = \bigsqcup_{\omega \in \Omega_l^{\theta+}} E_{\theta,\omega} T_{\xi(\theta)-\xi(\omega)}^{\omega_{|\omega|}^+}$$

where $E_{\theta,\omega} = f_{-\theta} f_\omega s^{\xi(\theta)-\xi(\omega)}$ is an isometry.

Proof. Writing $\theta = \theta_1\theta_2...\theta_k$ so that $|\theta| = k$, we have from the definitions

$$\begin{aligned}\Pi(\theta_1\theta_2...\theta_k) &= f_{-\theta_1\theta_2...\theta_k}\{\pi(\sigma) : \sigma \in \Omega^{\theta_k^-}_{\xi(\theta_1\theta_2...\theta_k)}\} \\ &= f_{-\theta_1\theta_2...\theta_k} s^{\xi(\theta_1\theta_2...\theta_k)} s^{-\xi(\theta_1\theta_2...\theta_k)}\{\pi(\sigma) : \sigma \in \Omega^{\theta_k^-}_{\xi(\theta_1\theta_2...\theta_k)}\} \\ &= E_{\theta_1\theta_2...\theta_k} T^{\theta_k^-}_{\xi(\theta_1\theta_2...\theta_k)}\end{aligned}$$

which demonstrates that $\Pi(\theta) = E_\theta T^{\theta^-_{|\theta|}}_{\xi(\theta)}$ where $E_\theta = f_{-\theta}s^{\xi(\theta)}$.

The last statement of the theorem follows similarly from Lemma 2. □

7. All tilings in $\mathbb{T}^\infty$ are quasiperiodic

We recall from [**16**] the following definitions. A subset P of a tiling T is called a *patch* of T if it is contained in a ball of finite radius. A tiling T is *quasiperiodic* if, for any patch P, there is a number $R > 0$ such that any disk centered at a point in the support of T and is of radius R contains an isometric copy of P. Two tilings are *locally isomorphic* if any patch in either tiling also appears in the other tiling. A tiling T is *self-similar* if there is a similitude ψ such that $\psi(t)$ is a union of tiles in T for all $t \in T$. Such a map ψ is called a *self-similarity*.

Theorem 10. *Let $(\mathcal{F}, \mathcal{G})$ be a tiling IFS.*

(1) *Each tiling in $\mathbb{T}_\infty$ is quasiperiodic.*
(2) *If $(\mathcal{F}, \mathcal{G})$ is coprime, then each pair of tilings in $\mathbb{T}_\infty$ are locally isomorphic.*
(3) *If $\theta \in \Sigma^\dagger_\infty$ is eventually periodic, then $\Pi(\theta)$ is self-similar. In fact, if $\theta = \alpha\overline{\beta}$ for some $\alpha, \beta \in \Sigma^\dagger_*$ then $f_{-\alpha}f_{-\beta}(f_{-\alpha})^{-1}\Pi(\theta)$ is a self-similarity.*

Proof. This uses Theorem 9, and follows similar lines to [**16**, proof of Theorem 2]. □

8. Addresses

Addresses, both relative and absolute, are described in [**16**] for the case $|\mathcal{V}| = 1$. See also [**6**]. Here we add information and generalize. The relationship between these two types of addresses is subtle and used in the proof of later Theorems.

8.1. Relative addresses.

Write $T^{(v)}_k$ to mean any of T^v_k or T_k.

Definition 9. The **relative address** of $t \in T^{(v)}_k$ is defined to be $\varnothing.\pi^{-1}s^k(t) \in \varnothing.\Omega^{(v)}_k$. The relative address of a tile $t \in T_k$ depends on its context, its location relative to T_k, and depends in particular on $k \in \mathbb{N}_0$. Relative addresses also apply to the tiles of $\Pi(\theta)$ for each $\theta \in \Sigma^\dagger_*$ because $\Pi(\theta) = E_\theta T^{\theta^-_{|\theta|}}_{\xi(\theta)}$ where $E_\theta = f_{-\theta}s^{\xi(\theta)}$ (by Theorem 9) is a known isometry applied to $T_{\xi(\theta)}$. Thus, the relative address of $t \in \Pi(\theta)$ (relative to $\Pi(\theta)$) is $\varnothing.\pi^{-1}f^{-1}_{-\theta}(t)$, for $\theta \in \Sigma^\dagger_*$.

Lemma 3. *The tiles of T_k are in bijective correspondence with the set of relative addresses $\varnothing.\Omega_k$. The tiles of T^v_k are in bijective correspondence with the set of relative addresses $\varnothing.\Omega^v_k$.*

Proof. We have $T_k = s^{-k}\pi(\Omega_k)$ so $s^{-k}\pi$ maps Ω_k onto T_k. Also the map $s^{-k}\pi : \Omega_k \to T_k$ is one-to-one: if $\beta \neq \gamma$, for $\beta, \gamma \in \Sigma_*$ then $f_\beta(A) \neq f_\gamma(A)$ because $t = s^{-k}\pi(\beta) = s^{-k}\pi(\gamma)$with $\beta, \gamma \in T_k$ implies $\beta = \gamma$. □

For precision we should write "the relative address of t relative to T_k" or equivalent: however, when the context $t \in T_k$ is clear, we may simply refer to "the relative address of t".

EXAMPLE 1. (Standard 1D binary tiling) For the IFS $\mathcal{F}_0 = \{\mathbb{R}; f_1, f_2\}$ with $f_1(x) = 0.5x, f_2(x) = 0.5x + 0.5$ we have $\Pi(\theta)$ for $\theta \in \Sigma_*^\dagger$ is a tiling by copies of the tile $t = [0, 0.5]$ whose union is an interval of length $2^{|\theta|}$ and is isometric to $T_{|\theta|}$ and represented by $tttt....t$ with relative addresses in order from left to right

$$\varnothing.111...11, \varnothing.111...12, \varnothing.111...21, ..., \varnothing.222...22,$$

the length of each string (address) being $|\theta|+1$. Notice that here T_k contains $2^{|\theta|}-1$ copies of T_0 (namely tt) where a copy is ET_0 where $E \in \mathcal{T}_{\mathcal{F}_0}$, the group of isometries generated by the functions of $\mathcal{F}_0$.

EXAMPLE 2. (Fibonacci 1D tilings) $\mathcal{F}_1 = \{ax, a^2x + 1 - a^2, a + a^2 = 1,\ a > 0\}$, $\mathcal{T} = \mathcal{T}_{\mathcal{F}_1}$ is the largest group of isometries generated by $\mathcal{F}_1$. The tiles of $\Pi(\theta)$ for $\theta \in \Sigma_*^\dagger$ are isometries that belong to $\mathcal{T}_{\mathcal{F}_1}$ (the group of isometries generated by the IFS) applied to the tiling of [0,1] provided by the IFS, writing the tiling T_0 as ls where l is a copy of $[0, a]$ and (here) s is a copy of $[0, a^2]$ we have:

$T_0 = ls$ has relative addresses $\varnothing.1, \varnothing.2$ (i.e. the address of l is 1 and of s is 2)
$T_1 = lsl$ has relative addresses $\varnothing.11, \varnothing.12, \varnothing.2$
$T_2 = lslls$ has relative addresses $\varnothing.111, \varnothing.112, \varnothing.12, \varnothing.21, \varnothing.22$
$T_3 = lsllslsl$ has relative addresses $\varnothing.1111, \varnothing.1112, \varnothing.112, \varnothing.121, ...$

We remark that T_k comprises F_{k+1} distinct tiles and contains exactly F_k copies (under maps of $\mathcal{T}_{\mathcal{F}_1}$) of T_0, where $\{F_k : k \in \mathbb{N}_0\}$ is a sequence of Fibonacci numbers $\{1, 2, 3, 5, 8, 13, 21, ...\}$.

Note that $T_4 = lsllslsllslls$ contains two "overlapping" copies of T_2.

8.2. Absolute addresses. The set of *absolute addresses* associated with $(\mathcal{F}, \mathcal{G})$ is

$$\mathbb{A} := \{\theta.\omega : \theta \in \Sigma_*^\dagger,\ \omega \in \Omega_{\xi(\theta)}^{\theta_{|\theta|}^-},\ \theta_{|\theta|} \neq \omega_1\}.$$

Define $\widehat{\Pi} : \mathbb{A} \to \{t \in T : T \in \mathbb{T}\}$ by

$$\widehat{\Pi}(\theta.\omega) = f_{-\theta}.f_\omega(A).$$

The condition $\theta_{|\theta|} \neq \omega_1$ is imposed. We say that $\theta.\omega$ is an *absolute address* of the tile $f_{-\theta}.f_\omega(A)$. It follows from Definition 5 that the map $\widehat{\Pi}$ is surjective: every tile of $\{t \in T : T \in \mathbb{T}\}$ possesses at least one absolute address.

In general a tile may have many different absolute addresses. The tile $[1, 1.5]$ of Example 1 has the two absolute addresses 1.21 and 21.211, and many others.

8.3. The Relationship between Relative and Absolute addresses.

THEOREM 11. *If $t \in \Pi(\theta) \backslash \Pi(\varnothing)$ with $\theta \in \Sigma_*^\dagger$ has relative address ω relative to $\Pi(\theta)$, then an absolute address of t is $\theta_1\theta_2...\theta_l.S^{|\theta|-l}\omega$ where $l \in \mathbb{N}$ is the unique index such that*

$$t \in \Pi(\theta_1\theta_2...\theta_l) \text{ and } t \notin \Pi(\theta_1\theta_2...\theta_{l-1}) \tag{8.1}$$

PROOF. Recalling that

$$\Pi(\theta|0) \subset \Pi(\theta_1) \subset \Pi(\theta_1\theta_2) \subset ... \subset \Pi(\theta_1\theta_2...\theta_{|\theta|-1}) \subset \Pi(\theta),$$

we have disjoint union

$$\Pi(\theta)=\Pi(\theta|0)\cup(\Pi(\theta_1)\backslash\Pi(\varnothing))\cup(\Pi(\theta_1\theta_2)\backslash\Pi(\theta_1))\cup...\cup\left(\Pi(\theta)\backslash\Pi(\theta_1\theta_2...\theta_{|\theta|-1})\right).$$

So there is a unique l such that Equation (8.1) is true. Since $t\in\Pi(\theta)$ has relative address ω relative to $\Pi(\theta)$ we have

$$\omega=\varnothing.\pi^{-1}f_{-\theta}^{-1}(t)$$

and so an absolute adddress of t is

$$\theta.\omega|_{cancel}=\theta.\pi^{-1}f_{-\theta}^{-1}(t)|_{cancel}$$

where $|_{cancel}$ means equal symbols on either side of "." are removed until there is a different symbol on either side. Since $t\in\Pi(\theta_1\theta_2...\theta_l)$ the terms $\theta_{l+1}\theta_{l+2}...\theta_{|\theta|}$ must cancel yielding the absolute address

$$\theta.\omega|_{cancel}=\theta_1\theta_2...\theta_l.\omega_{|\theta|-l+1}...\omega_{|\omega|}$$

□

9. Rigid tilings

9.1. Definitions. Let $\mathcal{U}$ be the group of all isometries on $\mathbb{R}^M$.

Definition 10. The family of tilings $\mathbb{T}:=\left\{\Pi(\theta):\theta\in\Sigma^{\dagger}\right\}$ and the tiling IFS $(\mathcal{F},\mathcal{G})$ are each said to be **rigid** when the following two statements are true: (i) if $E_1,E_2\in\mathcal{U}$ and $k\in\{0,1,...,a_{\max}-1\}$ are such that $E_1T_0^v\cap E_2s^kT_0^w$ tiles $E_1A^v\cap E_2s^kA^w$ then $E_1=E_2$, $k=0$, and $v=w$; (ii) if $E\in\mathcal{U}$ and $EA^v=A^w$ then $E=I$ and $v=w$.

A different notion of a rigid for the case $|\mathcal{V}|=1$ is in [**16**].

9.2. Inflation and deflation. We restrict attention to rigid tiling IFSs $(\mathcal{F},\mathcal{G})$.

Definition 11.

$$\begin{aligned}T_0^v&=\{\Pi(e|0):=f_e(A^{e^+}):e^-=v\}\\ \mathbb{Q}&:=\{ET:E\in\mathcal{U},T\in\mathbb{T}\}\\ \mathbb{Q}'&:=\{ET:E\in\mathcal{U},T\in\mathbb{T},\ T\neq T_0^v,v\in\mathcal{V}\}\end{aligned}$$

Definition 12. Let $(\mathcal{F},\mathcal{G})$ be a rigid tiling IFS. **Deflation** $\alpha:\mathbb{Q}'\to\mathbb{Q}$ is defined by $\alpha(T)=\{\alpha(t):t\in T\}$ where

$$\alpha(t):=\begin{cases} sEA_v & \text{if } t\in ET_0^v\subset T \text{ for some } E\in\mathcal{U},v\in\mathcal{V}\\ st & \text{otherwise}\end{cases}$$

for all $T\in\mathbb{Q}'$. ET_0^v is called the set of **partners** of $t\in ET_0^v$. If t_1 and t_2 are partners of t, then $\alpha(t_1)=\alpha(t_2)$. **Inflation** $\alpha^{-1}:\mathbb{Q}\to\mathbb{Q}'$ is defined by

$$\alpha^{-1}T=\{s^{-1}t:t\in T,t\text{ is not congruent to }sA_v\text{ for any }v\}\cup\{t\in sET_0^v:E\in\mathcal{U},sEA_v\in T\}$$

$$\alpha^{-1}(t):=\begin{cases} s^{-1} & \text{if } t\neq EsA_v \text{ for any } E\in\mathcal{U},v\in\mathcal{V}\\ ET_0^v & \text{if } t=EsA_v\end{cases}$$

for all $T\in\mathbb{Q}$. In particular, $\alpha T_k=T_{k-1}$ and $\alpha^{-1}T_{k-1}=T_k$ for all $k\in\mathbb{N}$.

It is straightfoward to show that inflation and deflation are well-defined on rigid tilings. See also [**16**, Lemmas 6 and 7]. It should be clear that rigidity is largely equivalent to recognizability [**1**] and to the unique composition property [**37**]. Rigidity extends these concepts to tiling IFSs.

LEMMA 4. *Let* $(\mathcal{F},\mathcal{G})$ *be a rigid tiling IFS. If* $\theta,\varphi \in \Sigma_*^\dagger$, $\Pi(\theta)\cap E\Pi(\varphi)\neq\emptyset$ *and* $\Pi(\theta)\cap E\Pi(\varphi)$ *tiles the intersection of the supports of* $\Pi(\theta)$ *and* $E\Pi(\varphi)$*, then*

$$\text{either } \Pi(\theta)\subset E\Pi(\varphi) \text{ or } E\Pi(\varphi)\subset \Pi(\theta)$$

PROOF. Follows at once from rigidity. □

9.3. Hierarchies of tilings.

THEOREM 12. *Let* $(\mathcal{F},\mathcal{G})$ *be a rigid tiling IFS. Let* T_k *be given.*
(i) The following ***hierarchy of*** $\sigma\in\Sigma_*$ *obtains:*

$$ET_0^{\sigma_{|\sigma|}^+}\subset F_1T_{\xi(S^{|\sigma|-1}\sigma)}^{\sigma_{|\sigma|}^+}\subset F_2T_{\xi(S^{|\sigma|-2}\sigma)}^{\sigma_{|\sigma|}^+}\subset ...\subset F_{|\sigma|-1}T_{\xi(S\sigma)}^{\sigma_{|\sigma|}^+}\subset T_{k=\xi(\sigma)}^{\sigma_{|\sigma|}^+} \tag{9.1}$$

where $F_j = s^{-\xi(S^{|\sigma|-j}\sigma)}E^{-1}_{\sigma_{|\sigma|-j}...\sigma_1}s^{\xi(S^{|\sigma|-j}\sigma)}$ *and* E_θ *is the isometry* $f_{-\theta}s^{\xi(\theta||\theta|)}$*. Application of* $\alpha^{\xi(\sigma_{|\sigma|})}$ *to the hierarchy of* $\sigma_1...\sigma_{|\sigma|}$ *minus the leftmost inclusion yields the heirarchy of* $\sigma_1...\sigma_{|\sigma|-1}$*.*
(ii) For all $\theta\in\Sigma_\infty^\dagger$, $n\in[N]$, $k\in\mathbb{N}_0$,

$$\alpha^{\xi(\theta|k)}E_{\theta|k}^{-1}\Pi(\theta)=\Pi(S^k\theta) \text{ and } \alpha^{-a_n}\Pi(\theta)=s^{-a_n}f_n\Pi(n\theta)$$

where $E_{\theta|k}=f_{-\theta|k}s^{\xi(\theta|k)}$*.*

PROOF. (i) Equation 9.1 is the result of applying $E^{-1}_{\sigma_{|\sigma|}\sigma_{|\sigma|-1}...\sigma_1}$ to the chain of inclusions

$$\Pi(\sigma_{|\sigma|}|0)\subset\Pi(\sigma_{|\sigma|})\subset\Pi(\sigma_{|\sigma|}\sigma_{|\sigma|-1})\subset...\subset\Pi(\sigma_{|\sigma|}\sigma_{|\sigma|-1}...\sigma_2)\subset\Pi(\sigma_{|\sigma|}\sigma_{|\sigma|-1}...\sigma_1)$$

where we recall that $\Pi(\theta)=E_\theta T_{\xi(\theta)}^{\theta_{|\theta|}^-}$ (Theorem 9) for all $\theta\in\Sigma_*^\dagger$, where $E_\theta := f_{-\theta}s^{\xi(\theta)}$.

(ii) This follows from $\Pi(\theta)=E_\theta T_{\xi(\theta)}^{\theta_{|\theta|}^-}$ and $\alpha T = sTs^{-1}\alpha$ for any $T:\mathbb{R}^M\to\mathbb{R}^M$. Taking $k=1$ in (ii) we have

$$\alpha^{a_{\theta_1}}\Pi(\theta)=s^{a_{\theta_1}}f_{\theta_1}^{-1}\Pi(S\theta) \text{ and } \alpha^{-a_n}\Pi(\theta)=s^{-a_n}f_n\Pi(n\theta). \tag{9.2}$$

□

Define for convenience:

$$\Lambda_k^v=\{\sigma\in\Sigma_*:\xi(\sigma)=k,\sigma_1^-=v\}\subset\Omega_{k-1}^v$$
$$\Lambda_k=\cup_v\Lambda_k^v\subset\Omega_{k-1}$$

THEOREM 13. *Let* $(\mathcal{F},\mathcal{G})$ *be a rigid tiling IFS. Let* T_k *be given.*
(i) There is a bijective correspondence between Λ_k^v *and the set of copies* $ET_0^v\subset T_k$ *with* $E\in\mathcal{T}$*.*
(ii) If $ET_0^v\subset T_k$ *for some* $E\in\mathcal{T}$*, then there is unique* $\sigma\in\Lambda_k^v$ *such that*

$$E=E^{-1}_{\sigma_{|\sigma|}\sigma_{|\sigma|-1}...\sigma_1}=(f_{-\sigma_{|\sigma|}\sigma_{|\sigma|-1}...\sigma_1}s^k)^{-1}=s^{-k}f_\sigma$$

PROOF. (i) If $(\mathcal{F},\mathcal{G})$ is rigid, then given either of the tilings ET_k or ET_k^v for some v and $E\in\mathcal{T}$ we can identify E uniquely.

The relative addresses of tiles in $ET_k=\cup ET_0^v$ may be calculated in tandem by repeated application of α^{-1} (see Section 9.2). Each tile in $ET_0=\cup ET_0^v$ is associated with a unique relative address in $\Omega_0=[N]$. Now assume that, for all $l=0,1,...,k$, $v\in\mathcal{V}$, we have identified the tiles of ET_l with their relative addresses (relative to T_l). These lie in $\Omega_l=\cup\Omega_l^v$. Then the relative addresses of the tiles of ET_{k+1} (relative to T_{k+1}) may be calculated from those of ET_k by constructing the set of sets $s^{-1}ET_k$, and then splitting the images of large tiles, namely those that are of the form $s^{-1}FA_v$ for some $v \in \mathcal{V}$ and $F \in \mathcal{U}$, to form nonintersecting sets of partners of the form $\{Ff_i(A^{i^+}) : i^- = v\}$, assigning to these "children of the split" the relative addresses of their parents (relative to T_k) together with an additional symbol $i \in [N]$ added on the right-hand end according to its relative address relative to the copy of T_0 to which it belongs. By rigidity, this can be done uniquely. The relative addresses (relative to T_{k+1}) of the tiles in $s^{-1}ET_k$ that are not split and so are simply s^{-1} times as large as their predecessors, are the same as the relative addresses of their predecessors relative to T_k.

(ii) It follows in particular that if $\mathcal{F}$ is locally rigid and $E'T_0^v \subset T_k$, then the relative addresses of the tiles of $E'T_0^v$ must be $\{\varnothing.\sigma_1...\sigma_{|\sigma|}i : i \in [N]\}$ for some $\sigma_1...\sigma_{|\sigma|} \in \Sigma_*^v$ with $\xi(\sigma_1...\sigma_{|\sigma|}) = k$. In this case we say that the relative address of $E'T_0^v$ (relative to T_k or T_k^v) is $\varnothing.\sigma_1...\sigma_{|\sigma|}$. □

9.4. A Key Theorem.

THEOREM 14. *Let* $(\mathcal{F}, \mathcal{G})$ *be a rigid tiling IFS. Then (i)* $\Pi : \Sigma^\dagger \to \mathbb{T}$ *is invertible; (ii)* $\Pi(\theta) = E\Pi(\psi)$ *for* $E \in \mathcal{U}$, *and* $\theta, \psi \in \Sigma_*^\dagger$, *if and only if are* $p, q \in \mathbb{N}_0$ *such that* $S^p\theta = S^q\theta$ *and* $E = f_{-\theta|p}(f_{-\psi|q})^{-1}$.

PROOF. Uses Lemma 4 and Theorem 13. See also [**17**], where the same result is proved for the case where the tiles have nonempty interiors. □

10. The big picture

General symbolic theory

Explanation of how tiling theory and study of fractal attractors are unified by IFS theory.

Explanation of the following diagram and associated figures.

$$\begin{array}{ccc} \Sigma \times \Sigma^\dagger & \overset{\mathbf{S}}{\longleftrightarrow} & \Sigma \times \Sigma^\dagger \\ \updownarrow & \pi \times \Pi & \updownarrow \\ A \times \mathbb{T} & \overset{\alpha}{\longleftrightarrow} & A \times \mathbb{T} \end{array}$$

measure theory too, and how to envisage and analyze spectral properties of $\mathbf{S}$

Pisot properties in the general (fractal) case

11. Relationship to Solomyak [36, 37]

unique composition property: combination of rotations as well as translations. Pisot number theorems via ergodic theorem applied to the big picture.

12. Relationship to Anderson&Putnam [1]

clarifies and generalizes the picture of the action of various euclidean groups and groupoids on e.g. $\mathbb{T}$, showing clearly what is actually going on

13. Relationship to Fast Basins, Fractal manifolds and Fractal Continuation

Commentary on relationships with earlier key structures identified by Barnsley and Vince. $\Pi(\theta)$ is a continuation of A while $\cup\Pi(\theta)$ is the fast basin. Examples and references. Fractal manifolds are made by gluing tilings together where they agree!

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