# Dynamics on Fractals

Michael F. Barnsley (MSI-ANU)

Lecture for **Symmetry in Newcastle**

1 November 2019

# Preliminary Remarks

- Thank you

# Preliminary Remarks

- Thank you
- Joint work with Louisa Barnsley, Andrew Vince

# plan

- (1) IFS, attractor A ,points $\pi(\sigma)$ and addresses $\sigma \in \Sigma$, golden b

# plan

- (1) IFS, attractor A ,points $\pi(\sigma)$ and addresses $\sigma \in \Sigma$, golden b
- (2) $\mathbb{T}$, tilings $\Pi(\theta)$, and addresses $\theta \in \Sigma^{\dagger}$

# plan

- (1) IFS, attractor A ,points $\pi(\sigma)$ and addresses $\sigma \in \Sigma$, golden b
- (2) $\mathbb{T}$, tilings $\Pi(\theta)$, and addresses $\theta \in \Sigma^{\dagger}$
- (3) $\pi \times \Pi : \Sigma \times \Sigma^{\dagger} \rightarrow A \times \mathbb{T}$

# plan

- (1) IFS, attractor A ,points $\pi(\sigma)$ and addresses $\sigma \in \Sigma$, golden b
- (2) $\mathbb{T}$, tilings $\Pi(\theta)$, and addresses $\theta \in \Sigma^{\dagger}$
- (3) $\pi \times \Pi : \Sigma \times \Sigma^{\dagger} \to A \times \mathbb{T}$
- (4) rigidity, recognizability, deflation $\alpha$ and inflation $\alpha^{-1}$

# plan

- (1) IFS, attractor A ,points $\pi(\sigma)$ and addresses $\sigma \in \Sigma$, golden b
- (2) $\mathbb{T}$, tilings $\Pi(\theta)$, and addresses $\theta \in \Sigma^{\dagger}$
- (3) $\pi \times \Pi : \Sigma \times \Sigma^{\dagger} \rightarrow A \times \mathbb{T}$
- (4) rigidity, recognizability, deflation $\alpha$ and inflation $\alpha^{-1}$
- (5) subshifts of finite type, graph IFS, and Markov measures

# plan

- (1) IFS, attractor A ,points $\pi(\sigma)$ and addresses $\sigma \in \Sigma$, golden b
- (2) $\mathbb{T}$, tilings $\Pi(\theta)$, and addresses $\theta \in \Sigma^{\dagger}$
- (3) $\pi \times \Pi : \Sigma \times \Sigma^{\dagger} \rightarrow A \times \mathbb{T}$
- (4) rigidity, recognizability, deflation $\alpha$ and inflation $\alpha^{-1}$
- (5) subshifts of finite type, graph IFS, and Markov measures
- (6) three examples, pedal triangle, golden bsquare, fish-horn

# plan

- (1) IFS, attractor A ,points $\pi(\sigma)$ and addresses $\sigma \in \Sigma$, golden b
- (2) $\mathbb{T}$, tilings $\Pi(\theta)$, and addresses $\theta \in \Sigma^{\dagger}$
- (3) $\pi \times \Pi : \Sigma \times \Sigma^{\dagger} \to A \times \mathbb{T}$
- (4) rigidity, recognizability, deflation $\alpha$ and inflation $\alpha^{-1}$
- (5) subshifts of finite type, graph IFS, and Markov measures
- (6) three examples, pedal triangle, golden bsquare, fish-horn
- (7) key theorem and action of the group of isometries

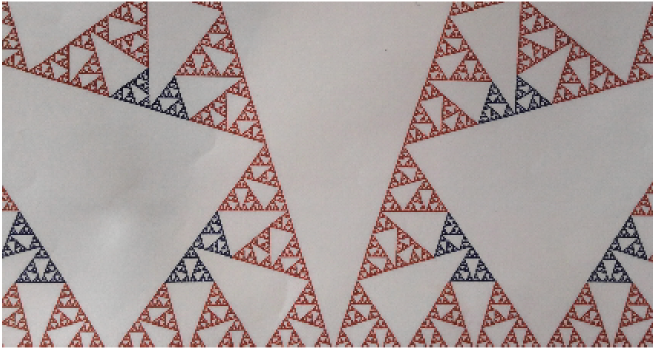
# plan

- (1) IFS, attractor A ,points $\pi(\sigma)$ and addresses $\sigma \in \Sigma$, golden b
- (2) $\mathbb{T}$, tilings $\Pi(\theta)$, and addresses $\theta \in \Sigma^{\dagger}$
- (3) $\pi \times \Pi : \Sigma \times \Sigma^{\dagger} \to A \times \mathbb{T}$
- (4) rigidity, recognizability, deflation $\alpha$ and inflation $\alpha^{-1}$
- (5) subshifts of finite type, graph IFS, and Markov measures
- (6) three examples, pedal triangle, golden bsquare, fish-horn
- (7) key theorem and action of the group of isometries
- (8) shift invariant measures and their relation to disjunctive points

# plan

- (1) IFS, attractor A ,points $\pi(\sigma)$ and addresses $\sigma \in \Sigma$, golden b
- (2) $\mathbb{T}$, tilings $\Pi(\theta)$, and addresses $\theta \in \Sigma^{\dagger}$
- (3) $\pi \times \Pi : \Sigma \times \Sigma^{\dagger} \rightarrow A \times \mathbb{T}$
- (4) rigidity, recognizability, deflation $\alpha$ and inflation $\alpha^{-1}$
- (5) subshifts of finite type, graph IFS, and Markov measures
- (6) three examples, pedal triangle, golden bsquare, fish-horn
- (7) key theorem and action of the group of isometries
- (8) shift invariant measures and their relation to disjunctive points
- (9) the big picture

example of a fractal tiling. see also abstract, monthly pape



# IFS, A, points and addresses, golden b

(1)

- IFS of similitudes

# IFS, A, points and addresses, golden b

(1)

- IFS of similitudes
- attractors and addresses: definition of $\pi : \Sigma \rightarrow 2^A$

# IFS, A, points and addresses, golden b

(1)

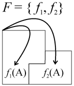
- IFS of similitudes
- attractors and addresses: definition of $\pi : \Sigma \rightarrow 2^A$
- OSC - components of attractors are non-overlapping

# IFS, A, points and addresses, golden b

(1)

- IFS of similitudes
- attractors and addresses: definition of $\pi : \Sigma \rightarrow 2^A$
- OSC - components of attractors are non-overlapping
- algebraic condition on scaling factors

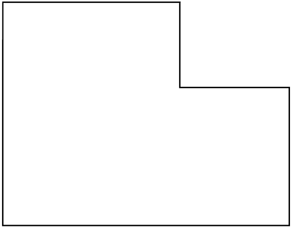
Golden b




A

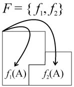

$$\sum_{\infty} = \{1,2\}^{\mathbb{N}} \qquad \sum_{*} = \bigcup_{k \in \mathbb{N}} \{1,2\}^{k}$$

π(1111121221222xxx)



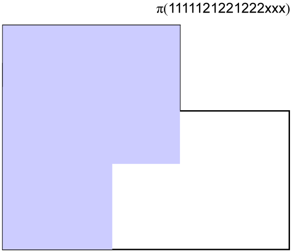
$$\pi : \sum_{\infty} \cup \sum_{*} \rightarrow A$$

# Golden b




A

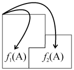

$$\sum_\infty = \{1,2\}^{\mathbb{N}} \qquad \sum_* = \bigcup_{k \in \mathbb{N}} \{1,2\}^k$$




$$\pi : \sum_\infty \cup \sum_* \rightarrow A$$

## Golden b

$F = \{f_1, f_2\}$

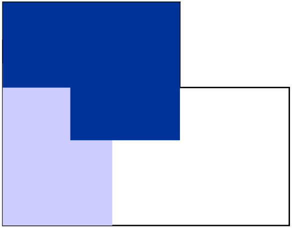



A

$G$


$\sum_{\infty} = \{1,2\}^{\mathbb{N}}$ $\quad \sum_{*} = \bigcup_{k \in \mathbb{N}} \{1,2\}^{k}$

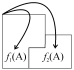
π(1111121221222xxx)



$$\pi : \sum_{\infty} \cup \sum_{*} \rightarrow A$$

## Golden b

$F = \{f_1, f_2\}$

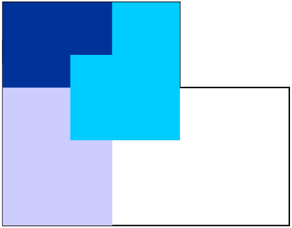



A

$G$


$\sum_\infty = \{1,2\}^{\mathbb{N}}$ $\sum_* = \bigcup_{k \in \mathbb{N}} \{1,2\}^k$

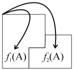
$\pi(1111121221222xxx)$



$$\pi : \sum_\infty \cup \sum_* \rightarrow A$$

## Golden b

$F = \{f_1, f_2\}$

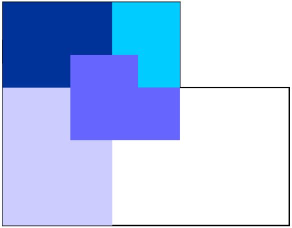



A

$G$

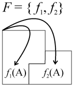

$\sum_\infty = \{1,2\}^{\mathbb{N}} \qquad \sum_* = \bigcup_{k \in \mathbb{N}} \{1,2\}^k$

π(1111121221222xxx)



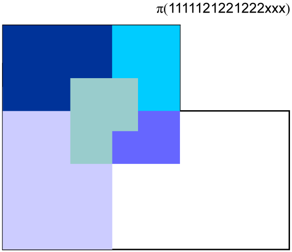
$$\pi : \sum_\infty \cup \sum_* \rightarrow A$$

## Golden b



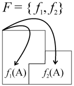

A


$\sum_{\infty} = \{1,2\}^{\mathbb{N}}$ $\sum_{*} = \bigcup_{k \in \mathbb{N}} \{1,2\}^{k}$



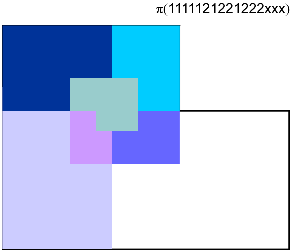

$$\pi : \sum_{\infty} \cup \sum_{*} \rightarrow A$$

# Golden b




A

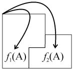

$\sum_{\infty} = \{1,2\}^{\mathbb{N}} \qquad \sum_{*} = \bigcup_{k \in \mathbb{N}} \{1,2\}^{k}$




$\pi : \sum_{\infty} \cup \sum_{*} \rightarrow A$

# Golden b

$F = \{f_1, f_2\}$

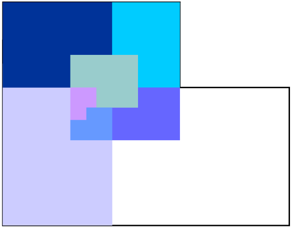



A

$G$

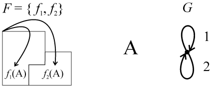

$\sum_\infty = \{1,2\}^{\mathbb{N}} \qquad \sum_* = \bigcup_{k \in \mathbb{N}} \{1,2\}^k$

π(1111121221222xxx)

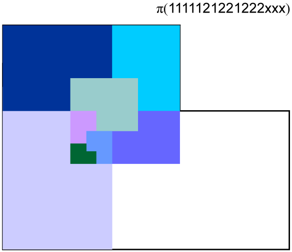


$\pi : \sum_\infty \cup \sum_* \longrightarrow A$

## Golden b

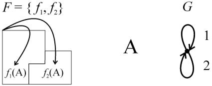



$$\sum_{\infty} = \{1,2\}^{\mathbb{N}} \qquad \sum_{*} = \bigcup_{k \in \mathbb{N}} \{1,2\}^{k}$$



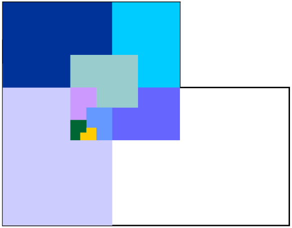

$$\pi : \sum_{\infty} \cup \sum_{*} \rightarrow A$$

# Golden b



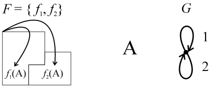

$\sum_\infty = \{1,2\}^{\mathbb{N}} \qquad \sum_* = \bigcup_{k\in\mathbb{N}} \{1,2\}^k$

π(1111121221222xxx)

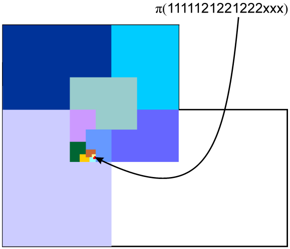


$$\pi : \sum_\infty \cup \sum_* \rightarrow A$$

## Golden b



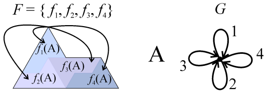

$$\textstyle\sum_\infty = \{1,2\}^{\mathbb{N}} \qquad \sum_* = \bigcup_{k\in\mathbb{N}}\{1,2\}^k$$



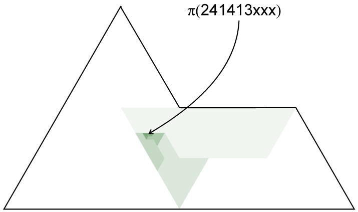

$$\pi : \textstyle\sum_\infty \cup \sum_* \to A$$

# Sphinx example

Sphinx




$$\sum_\infty = \{1,2,3,4\}^{\mathbb{N}} \quad \sum_* = \bigcup_{k\in\mathbb{N}} \{1,2,3,4\}^k$$




$$\pi : \sum_\infty \cup \sum_* \to A$$

# (2) How to construct tilings

$\Pi(\theta)$ for $\theta \in \Sigma$

- given $\theta \in \Sigma^{\dagger}$, how to construct $\Pi(\theta)$

# (2) How to construct tilings

$\Pi(\theta)$ for $\theta \in \Sigma$

- given $\theta \in \Sigma^{\dagger}$, how to construct $\Pi(\theta)$
- $\Pi(\theta|k) = f_{-\theta|k}\left\{\pi(\sigma) : \xi^{-}(\sigma) \leq \xi(\theta) < \xi(\sigma)\right\}$

# (2) How to construct tilings

$\Pi(\theta)$ for $\theta \in \Sigma$

- given $\theta \in \Sigma^{\dagger}$, how to construct $\Pi(\theta)$
- $\Pi(\theta|k) = f_{-\theta|k}\{\pi(\sigma) : \xi^{-}(\sigma) \leq \xi(\theta) < \xi(\sigma)\}$
- Theorem: $\Pi(\theta|0) \subset \Pi(\theta|1) \subset \Pi(\theta|2)...$

# (2) How to construct tilings

$\Pi(\theta)$ for $\theta \in \Sigma$

- given $\theta \in \Sigma^{\dagger}$, how to construct $\Pi(\theta)$
- $\Pi(\theta|k) = f_{-\theta|k}\{\pi(\sigma) : \xi^{-}(\sigma) \leq \xi(\theta) < \xi(\sigma)\}$
- Theorem: $\Pi(\theta|0) \subset \Pi(\theta|1) \subset \Pi(\theta|2)$...
- tiling metric

# (2) How to construct tilings

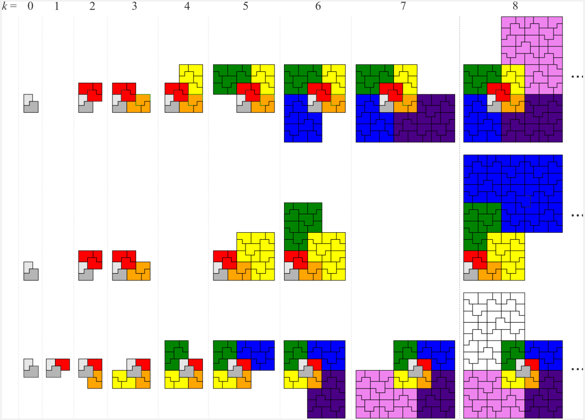
$\Pi(\theta)$ for $\theta \in \Sigma$

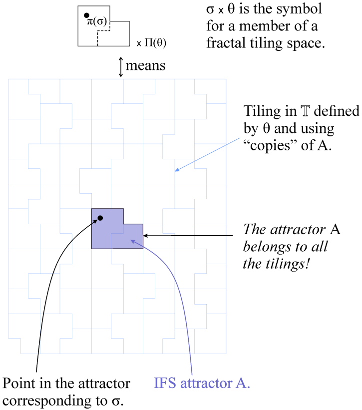
- given $\theta \in \Sigma^{\dagger}$, how to construct $\Pi(\theta)$
- $\Pi(\theta|k) = f_{-\theta|k}\{\pi(\sigma) : \xi^{-}(\sigma) \leq \xi(\theta) < \xi(\sigma)\}$
- Theorem: $\Pi(\theta|0) \subset \Pi(\theta|1) \subset \Pi(\theta|2)$...
- tiling metric
- illustrate for the golden b

# (2) How to construct tilings

$\Pi(\theta)$ for $\theta \in \Sigma$

- given $\theta \in \Sigma^{\dagger}$, how to construct $\Pi(\theta)$
- $\Pi(\theta|k) = f_{-\theta|k}\{\pi(\sigma) : \xi^{-}(\sigma) \leq \xi(\theta) < \xi(\sigma)\}$
- Theorem: $\Pi(\theta|0) \subset \Pi(\theta|1) \subset \Pi(\theta|2)$...
- tiling metric
- illustrate for the golden b
- applies to any OSC IFS


k = 0 1 2 3 4 5 6 7 8
...
...
...




$$\sigma \times \theta \ \in \ \Sigma \times \Sigma^{\dagger}$$

$$\pi(\sigma) \times \Pi(\theta) \ \in \ A \times \mathbb{T}$$

# (4) Rigidity, deflation and inflation

- the golden b is rigid

# (4) Rigidity, deflation and inflation

- the golden b is rigid
- deflation and inflation definitions

# (4) Rigidity, deflation and inflation

- the golden b is rigid
- deflation and inflation definitions
- (3) interplay with the big picture

# (5) Subshifts and definition of tiling IFS

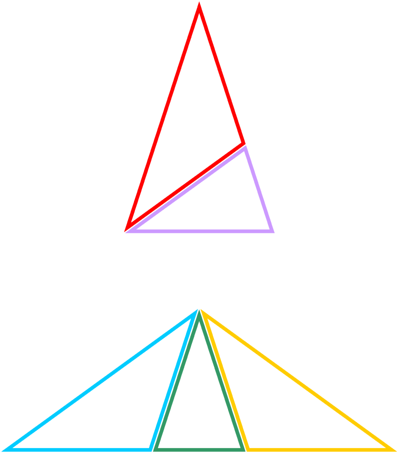
- subshifts of finite type, graph IFS, and Markov measures

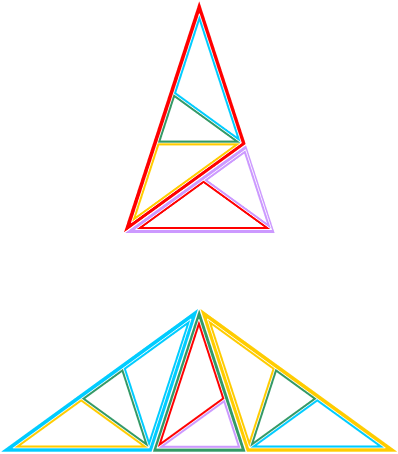
# (5) Subshifts and definition of tiling IFS

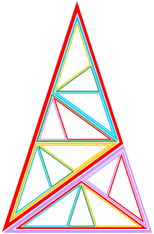
- subshifts of finite type, graph IFS, and Markov measures
- disjunctive points have measure zero and correspond to boundaries of tiles

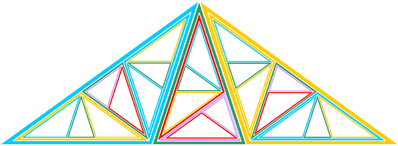
# (6) Examples

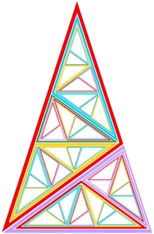
- robinson, golden bsquare, fish-horn, purely fractal

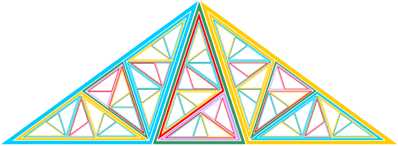


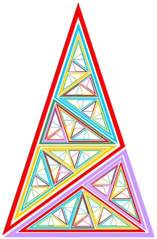


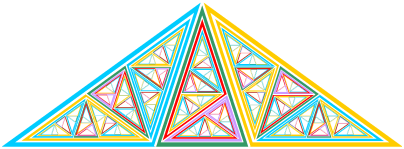


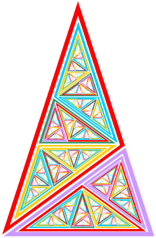


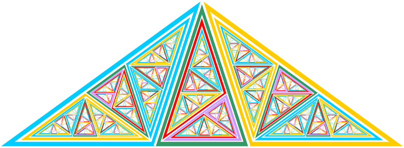


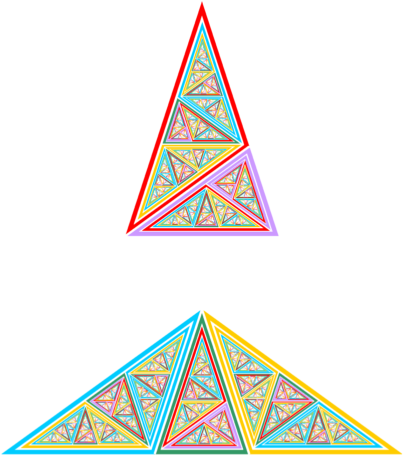












# (6) golden rectangle with golden b

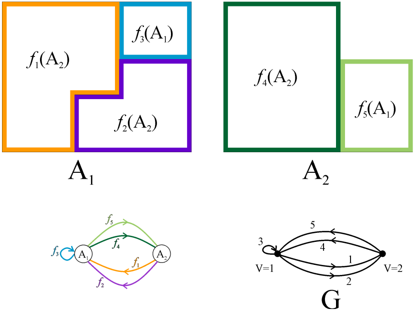
- built using 5 similitudes

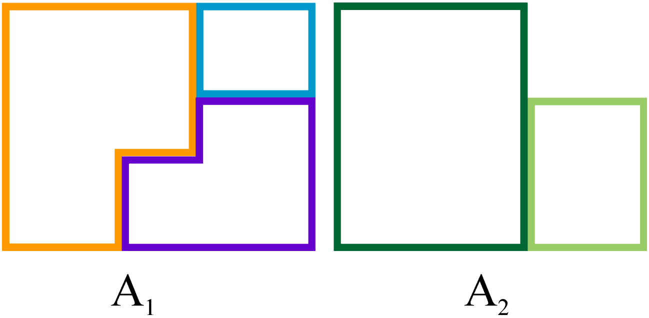
# (6) golden rectangle with golden b

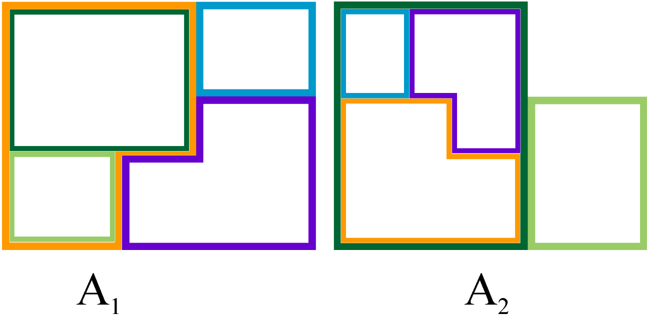
- built using 5 similitudes
- first describe the tiling IFS and the graph

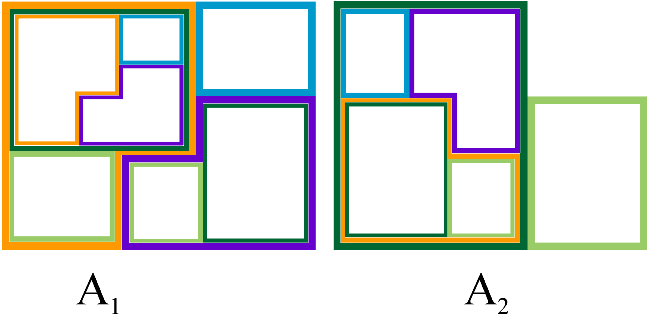
# (6) golden rectangle with golden b

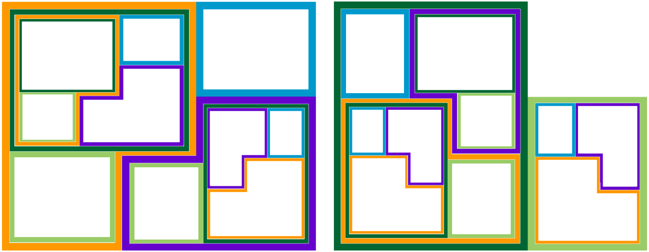
- built using 5 similitudes
- first describe the tiling IFS and the graph
- then illustrate the construction of the canonical tilings and of a tiling


$f_1(A_2)$
$f_3(A_1)$
$f_2(A_2)$
$f_4(A_2)$
$f_5(A_1)$
$A_1$
$A_2$
$f_5$
$f_4$
$f_3$
$A_1$
$A_2$
$f_1$
$f_2$
5
3
4
1
V=1
V=2
2
G

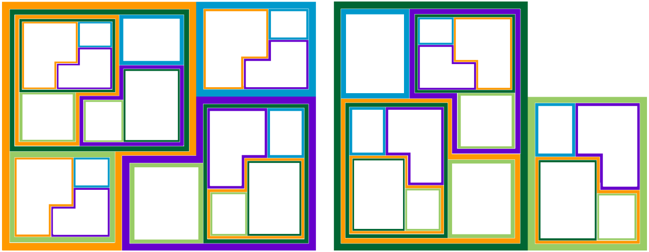

$A_1$
$A_2$


$A_1$
$A_2$

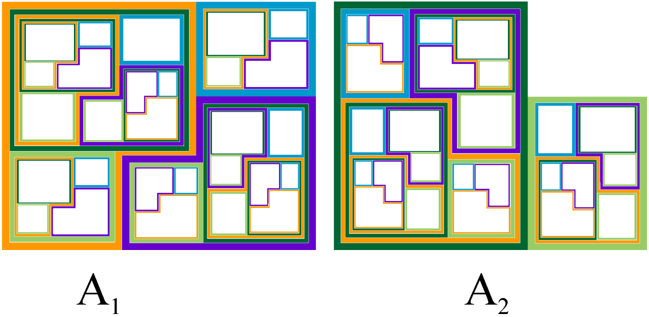

$A_1$
$A_2$

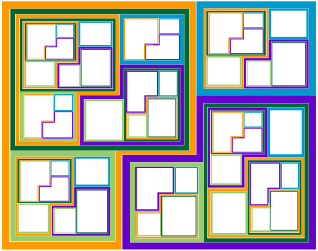


$A_1$ $A_2$

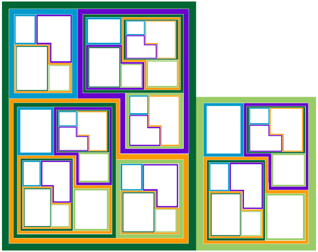


$A_1$ $A_2$

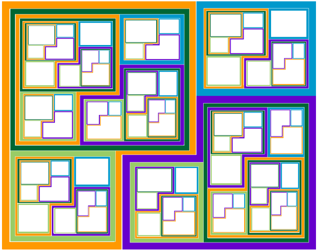

$A_1$
$A_2$



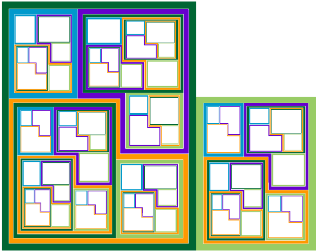
$A_1$



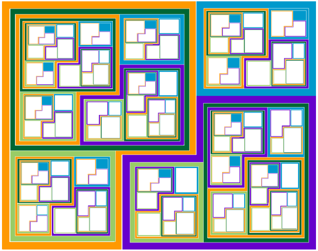
$A_2$



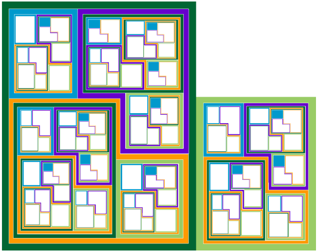
$A_1$



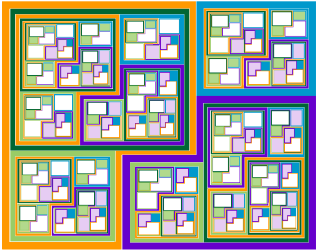
$A_2$



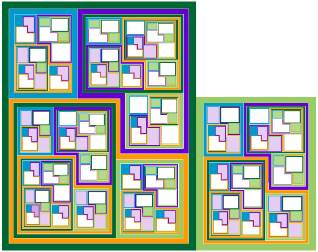
$A_1$



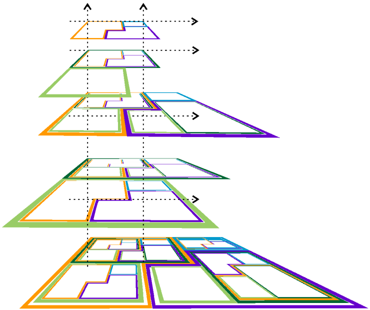
$A_2$



$A_1$



$A_2$



# (7) Main theorem

- key theorem and action of the group of isometries

# (7) Main theorem

- key theorem and action of the group of isometries
- canonical tilings

# (7) Main theorem

- key theorem and action of the group of isometries
- canonical tilings
- beautiful combinatorial formula explains all

# (8 and 9) Big picture

- the big picture including dynamics

# (8 and 9) Big picture

- the big picture including dynamics
- local isomorphism, self-similar tiling theory vs tiling IFS theory

# (8 and 9) Big picture

- the big picture including dynamics
- local isomorphism, self-similar tiling theory vs tiling IFS theory
- Anderson and Putnam, Solomyak

# (8 and 9) Big picture

- the big picture including dynamics
- local isomorphism, self-similar tiling theory vs tiling IFS theory
- Anderson and Putnam, Solomyak
- the importance of boundaries

# (10) main points I care about

- IFS tiling theory unifies, simplifies and extends s.s. tiling

# (10) main points I care about

- IFS tiling theory unifies, simplifies and extends s.s. tiling
- big picture of shift dynamics

# (10) main points I care about

- IFS tiling theory unifies, simplifies and extends s.s. tiling
- big picture of shift dynamics
- key theorem and its significance

# (10) main points I care about

- IFS tiling theory unifies, simplifies and extends s.s. tiling
- big picture of shift dynamics
- key theorem and its significance
- symbolic theory $\leftrightarrow$ geometry

# (10) main points I care about

- IFS tiling theory unifies, simplifies and extends s.s. tiling
- big picture of shift dynamics
- key theorem and its significance
- symbolic theory $\leftrightarrow$ geometry
- understanding canonical tilings is key