Coxeter systems for which the Brink-Howlett automaton is minimal.

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COXETER SYSTEMS

- Recall: A **Coxeter System** is a pair (W, S) consisting of a group W and a set of generators $S \subseteq W$ subject only to relations of the form

$$(st)^{m(s,t)} = 1$$

where $m(s,s) = 1$ and $m(t,s) = m(s,t) \geq 2$ for $s \neq t$. ($m(s,t) = \infty$ is allowed).

EXAMPLES

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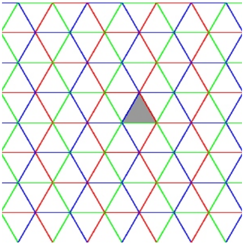
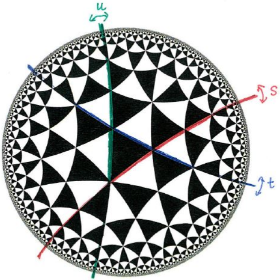
- Weyl groups of simple Lie algebras
- Triangle groups corresponding to tessellations of the Euclidean/Hyperbolic plane.





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  Since $sts = tst$ and $utu = tut$. We have

$$stsutu = tsttut = tsut$$

Coxeter Graphs

Coxeter graph of (W, S): vertices labelled by $s \in S$ and there is an edge between vertices s and t if and only if $m(s, t) \geq 3$. The edge is labelled only if $m(s, t) > 3$.

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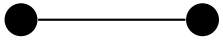

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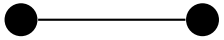

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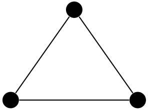

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- Given a string of generators, is it a reduced expression?

AUTOMATA FOR GROUPS

Definition

Let W be a group with generating set S. A **Finite State Automaton** *for $(W;S)$ is a finite directed graph capable of reading words $w \in W$ and giving the answer YES if and only if the word w is reduced.*

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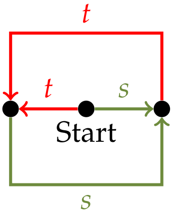
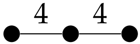


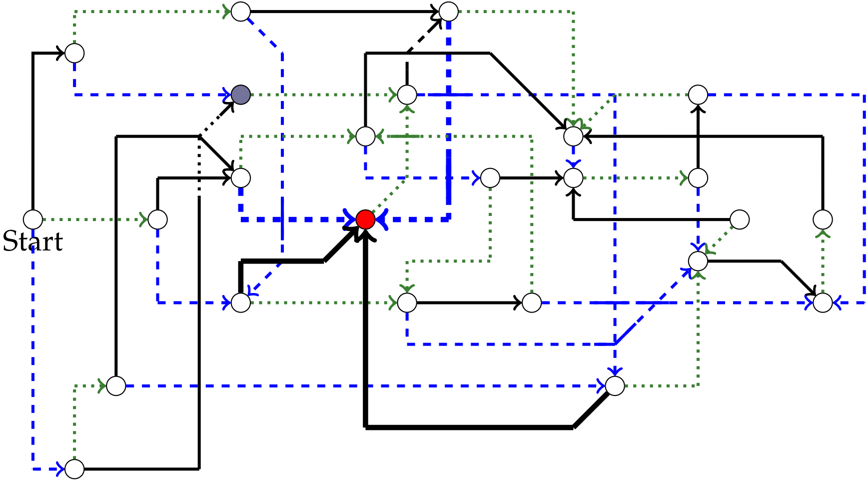
Finite State Automata for Coxeter Groups

Theorem (Brink-Howlett, 1993)

For each finitely generated Coxeter group W, there exists a finite state automaton which recognises the language of reduced words of W.

BRINK-HOWLETT AUTOMATON FOR $\tilde{B}_2$





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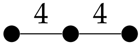

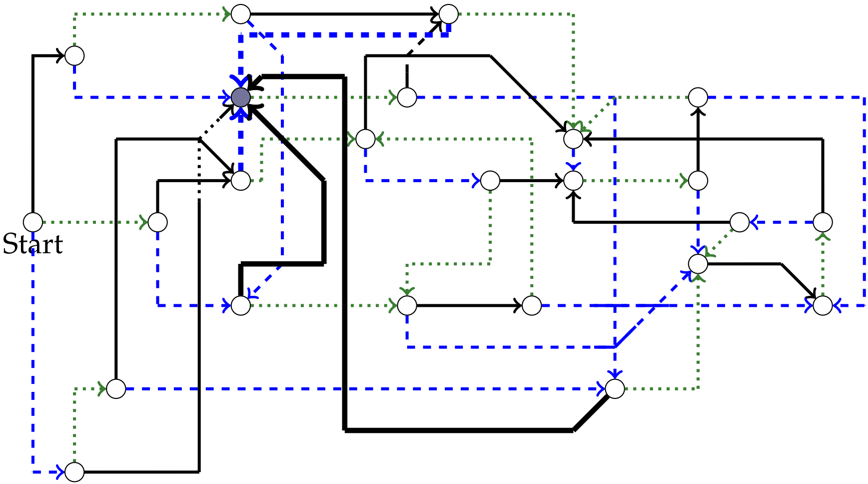

For which Coxeter systems is the Brink-Howlett automaton minimal?

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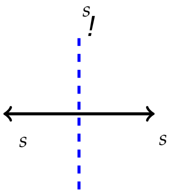

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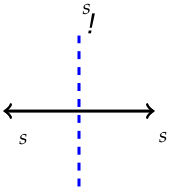

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- **Remark:** This is a faithful representation.

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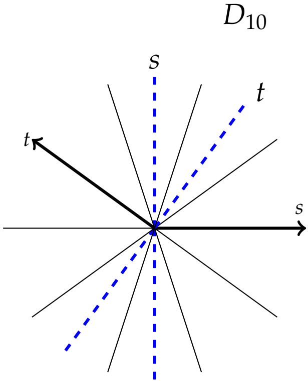

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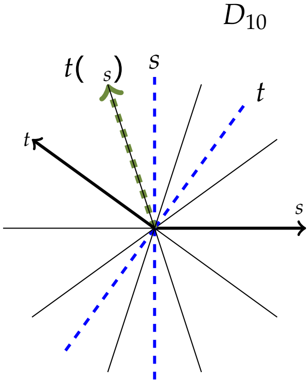

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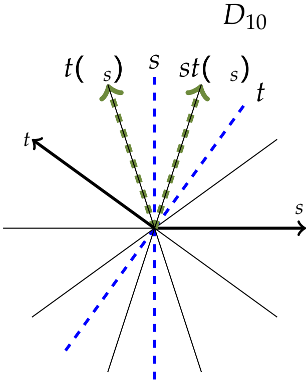

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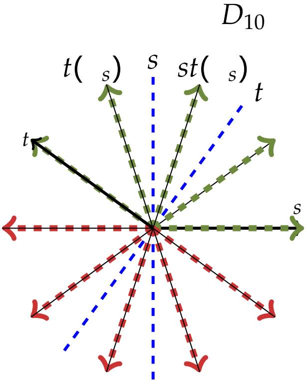

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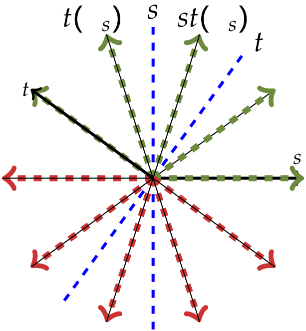

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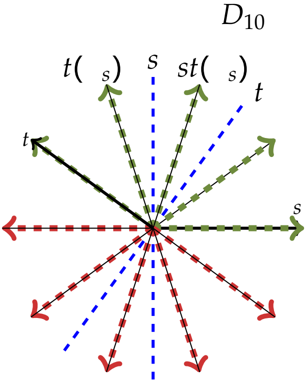
- $\{\ _s \mid s \in S\}$ are the **simple roots**.
- Any root $\ \in\ $ is either a **positive** or **negative** linear combination of the basis of simple roots.



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- Given a reduced expression for $w \in W$ and $s \in S$ we want to know:
- Whether $\ell(ws) > \ell(w)$
- Where to direct the edge s from a state representing w to the state representing ws.

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- Let $\Phi(w) = \{\beta \in \Phi^+ \mid w(\beta) < 0\}$. $\Phi(w)$ is called the *inversion set* of w.

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- If $_s \notin \quad (w)$ then $\ell(ws) > \ell(w)$ and

$$(ws) = \{ \quad _s\} \cup s\{ \quad (w)\}$$

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$$\mathcal{E}(w) := \Phi(w) \cap \mathcal{E}$$

- If $\alpha_s \notin \mathcal{E}(w)$ then $\ell(ws) > \ell(w)$ and

$$\mathcal{E}(ws) = (\{\alpha_s\} \cup s\{\mathcal{E}(w)\}) \cap \mathcal{E}$$

Conjecture (Hohlweg-Nadeau-Williams, 2016)

The Brink-Howlett automaton A_{BH} is minimal if and only if

$$E = {}^{+}_{\mathrm{sph}}.$$

*where ${}^{+}_{\mathrm{sph}}$ is the set of positive roots whose support generates a finite Coxeter group (called **spherical roots**).*

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Conjecture (Hohlweg-Nadeau-Williams, 2016)

The Brink-Howlett automaton $\mathcal{A}_{BH}$ is minimal if and only if

$$E = \Phi^+_{sph}.$$

where Φ^+_{sph} is the set of positive roots whose support generates a finite Coxeter group (called *spherical roots*).

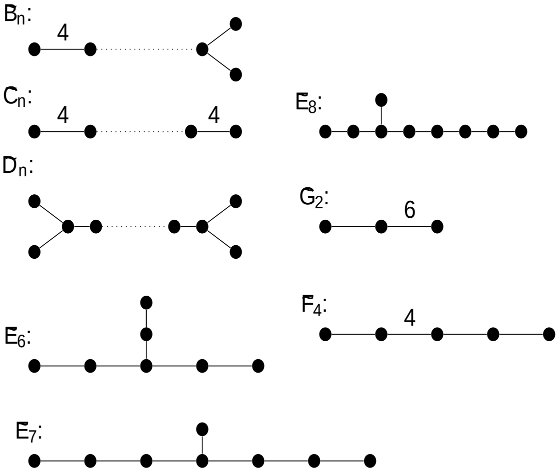
- For $\beta \in \Phi^+$, can write $\beta = \sum_{s \in S} c_s \alpha_s$ with $c_s \geq 0$.
- The *support* of $\beta \in \Phi$ is the set $J(\beta) = \{s \in S \mid c_s \neq 0\}$. Eg. if $\beta = \alpha_s + \alpha_t$ then $J(\beta) = \{s, t\}$.

Define X to be the following set of Coxeter graphs:

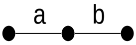
$$X = \{\text{affine irreducible}\} \bigcup \{\text{compact hyperbolic}\} :$$

with no circuits or infinite bonds.

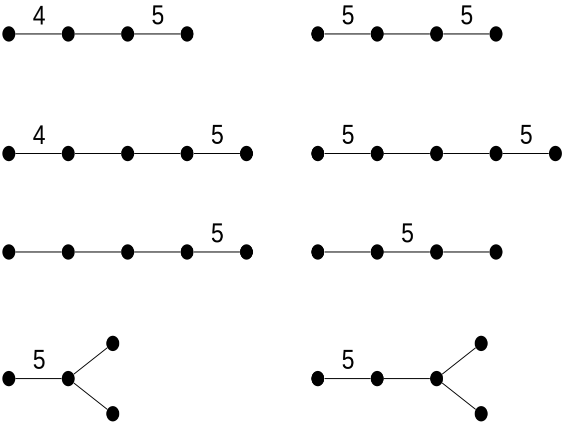
Affine irreducible graphs (other than $\tilde{A}_n$)



Compact hyperbolic graphs with no curcuits or in nite bonds



where $a; b < 1$, $\frac{1}{a} + \frac{1}{b} < \frac{1}{2}$.



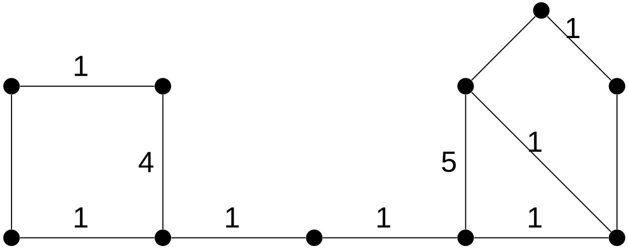
Theorem (J. Parkinson, Y.Y, 2018)

Let $(W; S)$ be a nitely generated Coxeter system. The following are equivalent:

(1) The Brink-Howlett automaton A_{BH} is minimal.

(2) The Coxeter graph of $(W; S)$ does not have a subgraph contained in X .

(3) The set of elementary roots E is $= {}^{+}_{sph}$.



The automaton A_{BH} is minimal for this Coxeter group!

MINIMAL AUTOMATON

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$$T(w) = \{v \in W \mid \ell(wv) = \ell(w) + \ell(v)\}.$$

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- The automaton $\mathcal{A}_{BH}$ is minimal if and only if $T(w) = T(v)$ whenever $\mathcal{E}(w) = \mathcal{E}(v)$.

OUTLINE OF PROOF

(1) =) (2): If A_{BH} is minimal for W then $_W$ does not have a subgraph in X .

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(1) $\Rightarrow$ (2): If A_{BH} is minimal for W then $_W$ does not have a subgraph in X.

Lemma

Let (W, S) be a finitely generated Coxeter system. If there exists $J \subseteq S$ and $t \in S$ such that:

(i) J is spherical, and

(ii) $J \cup \{t\}$ is not spherical, and

(iii) $w_J(\ _t) \in E$, where w_J is the unique longest element of W_J.

Then $T(t w_J) = T(w_J)$ and $E(w_J) \neq E(t w_J)$.

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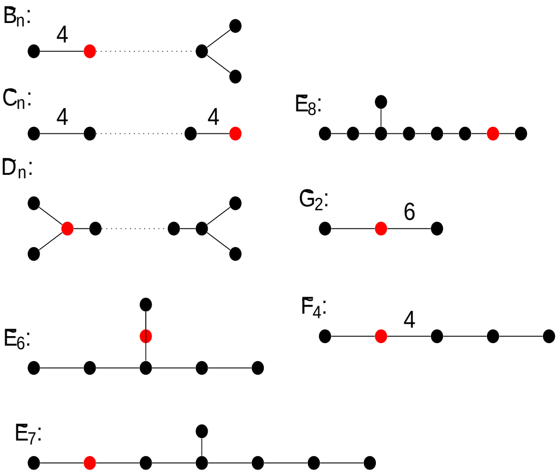
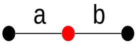
- Fact: Let ' be the highest root of $_0$. There is a unique simple root $_t$, such that $\langle ';\ _t \rangle = 1$ and $\langle ';\ _i \rangle = 0$ for all other simple roots $_i$.
- Let $t \in S$ be the simple reflection associated to $_t$.



Take t to be the red dot and $J = S \setminus \{t\}$. Then $T(tw_J) = T(w_J)$ and $E(tw_J) \neq E(w_J)$.

For compact hyperbolic graphs, let t be the red dot. Then $J = S \setminus \{t\}$.



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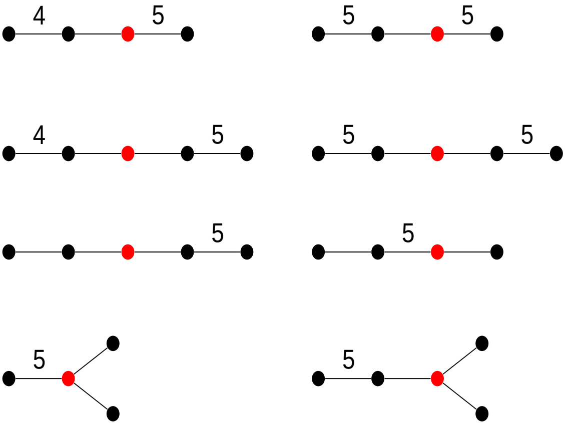

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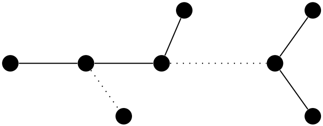

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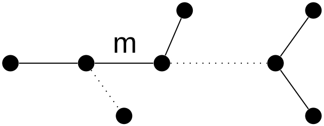

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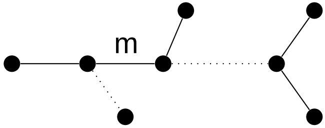
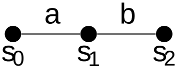


- Nonexistence of sub-graphs of type G_2 and



where $a; b < 1 , \frac{1}{a} + \frac{1}{b} < \frac{1}{2}$.

=) $m < 6$.

- Suppose $m = 5$.

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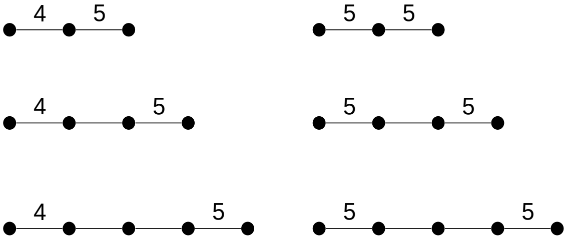
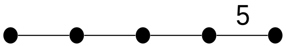


$\Longrightarrow \quad d(e_m, e_j) > 3$.

- However, nonexistence of



gives a contradiction.

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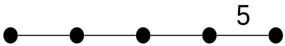
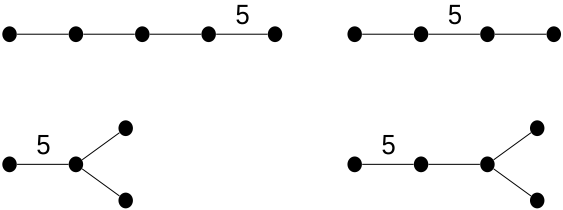


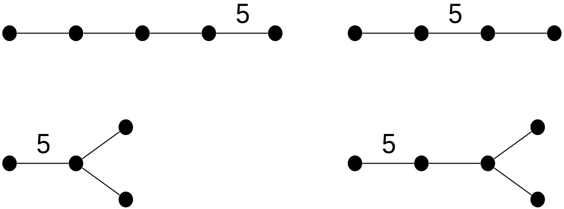
  gives a contradiction.

- Therefore, there is a unique edge label of $m = 5$.

- But nonexistence of

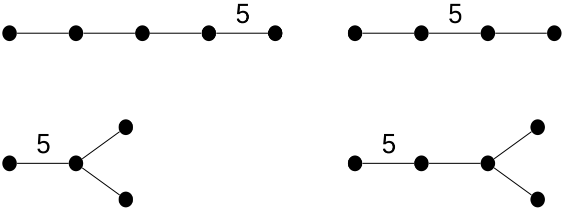


- But nonexistence of



- =) (J( )) must be of type H_3 or H_4.

- But nonexistence of



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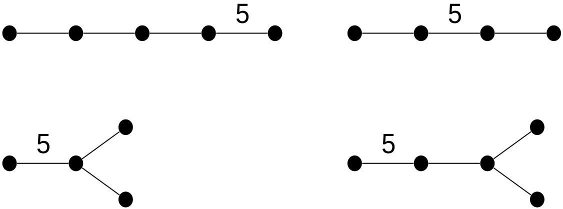

- =) (J( )) must be of type H_3 or H_4.

- But these are nite Coxeter groups. Contradiction.
- Casesm = 4 and m = 3 are similar.

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- Using the key fact that A_{BH} is minimal for nite Coxeter groups.

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- Hence good reasons to explore more of this story...

Thank you.