Neretin groups admit no nontrivial invariant random subgroups

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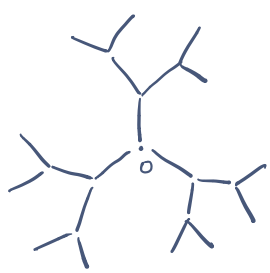
1. Background on Neretin groups

In the 1990s Neretin introduced a class of groups as a combinatorial analogue of the diffeomorphism group of the circle.

Let T be a regular tree of finite degree q, $q \geq 3$.

T



∂T = the boundary of T

(identified as infinite geodesic rays from o)

$\mathrm{Aut}(T)$ = the automorphism group of the tree T.

Equip $\mathrm{Aut}(T)$ with the topology of pointwise convergence.

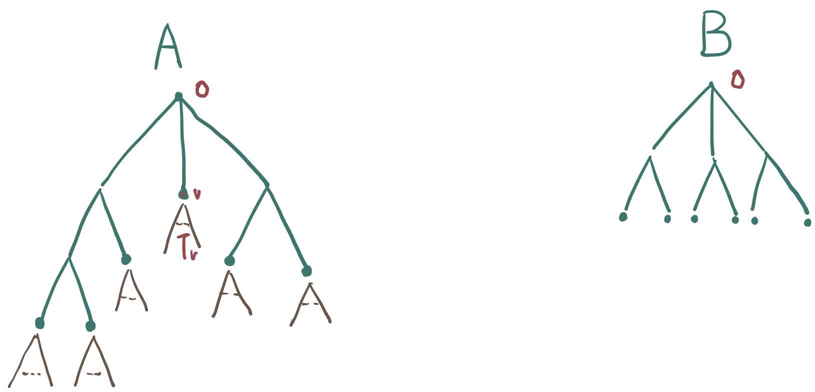
The Neretin group N_q is the topological full group of $\mathrm{Aut}(T) \curvearrowright \partial T$.

That is, a homeomorphism $g \in \mathrm{Homeo}(\partial T)$ is in N_q, iff one can find a partition of ∂T into disjoint clopen subsets $\partial T = \bigsqcup_{i=1}^{k} U_i$ s.t. for each $i \in \{1,2,\dots,k\}$, $\exists\, f_i \in \mathrm{Aut}(T)$,

$$g\big|_{U_i} = f_i\big|_{U_i}.$$

Explicitly, $g \in N_q$ can be represented by a triple (A, B, φ), where A, B are complete finite subtrees, $|\partial A| = |\partial B|$, φ is forest isomorphism

Ex:



φ can be written as $\varphi = \sigma(\varphi_v)_{v\in\partial A}$,

$\sigma: \partial A \to \partial B$ a bijection, φ_v is a rooted automorphism in T_v.

Such homeomorphisms are also called

- Spheromorphisms of ∂T (Neretin)
- almost automorphism of T
- near automorphism of T (Cornulier)
- germs of automorphisms of $\mathrm{Aut}(T)$ (Caprace - de Medts)

Simple locally compact groups acting on trees and their germs of automorphisms

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Some properties of N_q

- N_q carries a group topology s.t. the natural inclusion
$$\mathrm{Aut}(T) \hookrightarrow N_q$$
is continuous and open; w.r.t. this topology, N_q is a non-discrete t.d.l.c. group.

- The Higman-Thompson group $V_{q-1,q}$ embeds as a dense subgroup of N_q. As a consequence, N_q is compactly generated.

- (Kapoudjian 99) N_q is abstractly simple.

2. Invariant random subgroups

Let G be a locally compact second countable group. Consider its Chabauty space.

$\mathrm{SUB}(G)$ = the set of closed subgroups of G.

The Chabauty topology is generated by open sets of the form

$\mathcal{O}_1(K) = \{H \in \mathrm{SUB}(G) : H \cap K = \emptyset\}$, $K \subset G$ compact

$\mathcal{O}_2(U) = \{H \in \mathrm{SUB}(G) : H \cap U \neq \emptyset\}$, $U \subset G$ open.

G acts on $\mathrm{SUB}(G)$ by conjugation.

An invariant random subgroup of G is a G-invariant Borel probability measure on $\mathrm{SUB}(G)$.

Prop. (Abért - Bergeron - Biringer - Gelander - Nikolov - Raimbault - Samet)

Any IRS μ of G is induced by some probability measure preservin action of G.

$G \curvearrowright (X, \nu)$, ν preserved by G.

Consider the map
$$X \to \mathrm{SUB}(G)$$
$$x \mapsto \mathrm{stab}_G(x) = \{g \in G : g.x = x\}$$

Then the pushforward of ν is an IRS of G.

Examples of IRSs :

- $\delta_{\{N\}}$, $\quad N \trianglelefteq G$ normal subgroup

  Trivial IRSs refer to $\delta_{\{\{id\}\}}$, $\delta_{\{G\}}$.

- $\Gamma \leq G$ closed subgroup of finite covolume

$$\begin{aligned} G &\to SUB(G) \\ g &\mapsto g\Gamma g^{-1} \end{aligned} \qquad \text{factors through } G/\Gamma.$$

  That is, the normalized Haar measure on G/Γ gives an IRS supported on conjugates of Γ.

- G acts on a rooted tree T_o by automorphisms

$$G \curvearrowright (\partial T_o, m), \quad m \text{ invariant measure}$$

  It gives rise to a stabilizer IRS, supported on

$$\{ Stab_G(x) \}_{x \in \partial T_o}.$$

Expect to have more IRSs :

choose a fixed point set $C \subset \partial T$.

and take $\mathrm{Fix}_G(C) = \{g \in G : g.x = x \text{ for all } x \in C\}$

$$\mathcal{F}(\partial T) = \{\text{closed subsets of } \partial T\} \to \mathrm{SUB}(G)$$
$$C \mapsto \mathrm{Fix}_G(C)$$

push forward a G-invariant measure on $\mathcal{F}(\partial T)$, we get an IRS.

Evidence supporting there is no interesting IRSs

Say G is defined by its action $G \curvearrowright X$.

Suppose we already know $G \curvearrowright \mathcal{F}(X)$ admits no ergodic invariant measure other than $\delta_{\{\emptyset\}}$, $\delta_{\{X\}}$.

Example : The Higman-Thompson group $V_{d,k}$ acts on $\partial T_{d,k}$, there is no invariant measure on $\mathcal{F}(\partial T_{d,k})$ other than the trivial ones.

In the situation of countable topological full groups,

one can upgrade

no nontrivial ergodic invariant measure on $\mathcal{F}(X)$

$\to$ no nontrivial ergodic IRS

3. Double commutator lemma for IRS (Z, 19)

Let Γ be a countable group acting faithfully on a second countable Hausdorff space X by homeomorphisms.

Let μ be an ergodic IRS of Γ, $\mu \neq \delta_{\{id\}}$.

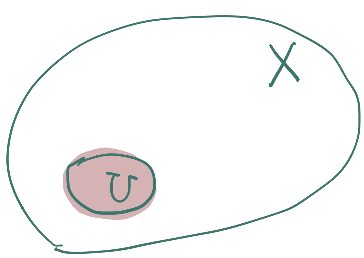
Then for μ-a.e. H, there exists $U \subset X$, U open and non-empty, s.t.

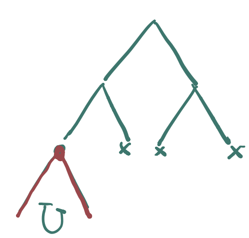
$$H \geq [R_\Gamma(U), R_\Gamma(U)].$$

$\mathrm{Rist}_\Gamma(U)$

$R_\Gamma(U)$ is the rigid stabilizer of Γ in U,

$$R_\Gamma(U) = \{ g \in \Gamma : g.x = x \text{ for all } x \in U^c \}$$





Lemma Continued

If in addition, we assume that $R_\Gamma(U)$ has no fixed point in U, for all non-empty open U.

Then for μ-a.e. H, if $x \in X$ is not a fixed pt. of H, then there is an open nbhd V of x, s.t.

$$H > [R_\Gamma(V), R_\Gamma(V)].$$

The double commutator lemma for normal subgroup is well known,

Higman simplicity thm, Grigorchuk, Matui, Nekrashevych, ...

For IRSs it is used in a similar way to relate H to its fixed point set on X.

Example : The derived subgroup $V'_{d,k}$ has no nontrivial IRSs (Dudko - Medynets).

Can the lemma be extended to t.d.l.c. groups?

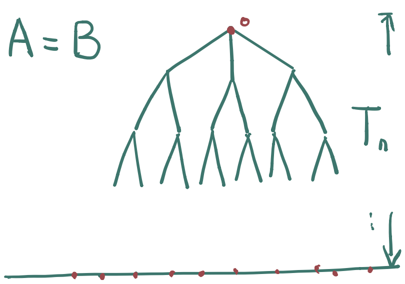
By explicit counting arguments, shown to be true in elliptic subgroups of Neretin group $\mathcal{N}_q$ (Z. 19)

Bader - Caprace - Gelander - Mozes : $\mathcal{N}_q$ does not contain any lattice

Proof goes through open subgroup $\mathcal{O}$:

expand the tree to level n

$A = B$



$$\mathcal{O}_n = \{(A, B, \varphi) : A = B = T_n\}$$

$$= \Big(\prod_{v \in L_n} \mathrm{Aut}(T_v)\Big) \rtimes \mathrm{Sym}(L_n)$$

$$\mathcal{O} = \bigcup \mathcal{O}_n$$

Suppose Γ is a lattice in $\mathcal{N}_q$.

then $\Gamma_0 = \Gamma \cap \mathcal{O}$ is a lattice in $\mathcal{O}$.

Contradiction comes from

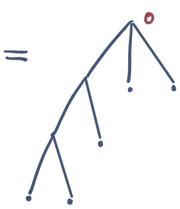
- Γ_0 is discrete $\rightsquigarrow$ $\exists\, n$ large enough, $\Gamma \cap \Big(\prod_{v \in L_n} \mathrm{Aut}(T_v)\Big) = \{id\}$.
- Γ_0 has finite covolume in $\mathcal{O}$ $\rightsquigarrow$ the projection of

$\Gamma \cap \mathcal{O}_k$ to $\mathrm{Sym}(\mathcal{L}_k)$ satisfies a specific volume lower bnd.

4. Neretin groups have no nontrivial IRS

$$\mathcal{O}_A = \bigcup_{n=0}^{\infty} \mathcal{O}_{A,n}$$

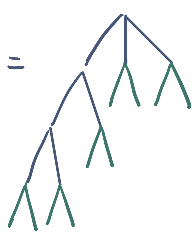
e.g. $A =$



A_n = expand A down n levels

$$\mathcal{O}_{A,n} = \left(\prod_{u \in \partial A_n} \mathrm{Aut}(T_u) \right) \rtimes \mathrm{Sym}(\partial A_n)$$

$A_1 =$



Outline: Let μ be an ergodic IRS of $\mathcal{N}_q$, $\mu \neq \delta_{\{id\}}$.

1. H has to intersect some $\mathcal{O}_A$

Lemma: for μ-a.e. H, there exists a finite complete A,

s.t. $H \cap \mathcal{O}_A \neq \{id\}$

Step 1 implies it is meaningful to consider induced IRSs

$$\mathrm{SUB}(N_q) \to \mathrm{SUB}(\mathcal{O}_A)$$
$$H \mapsto H \cap \mathcal{O}_A$$

push μ forward to μ_A,

where A goes over finite complete subtrees (countable collection)

2. Consider an ergodic IRS η of $\mathcal{O}_A$, $\eta \neq \delta_{\{id\}}$.

Prop: for η-a.e. H, there exists $U \subset \partial T$, U open, non-empty, such that

$$H \geq [R_{\mathcal{O}_A}(U), R_{\mathcal{O}_A}(U)].$$

3. Recall that the Higman-Thompson group $V_{q-1,q}$ is dense in N_q. It follows that if H contains $[R_{\mathcal{O}_A}(U), R_{\mathcal{O}_A}(U)]$, then $H \cap V' \neq \{id\}$.

1 + 2 implies that the induced IRS

$$\mathrm{SUB}(N_q) \to \mathrm{SUB}(V')$$
$$H \mapsto H \cap V'$$

μ pushforward to $\mu_{V'}$,

does not charge the trivial subgroup $\{id\}$.

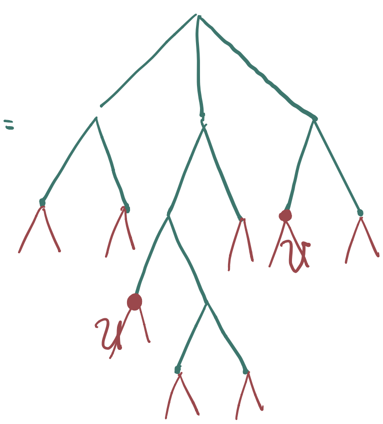
Finally, invoke the result that V' has no nontrivial IRSs, it follows $\mu_{V'} = \delta_{\{V'\}}$.

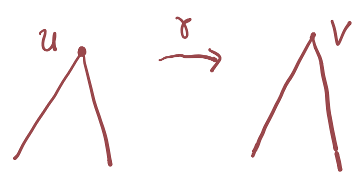
Back to $\mathrm{SUB}(N_q)$, it means μ-a.e. $H \geq V'$.

Key picture in step 2:

Covering, intersection, inducing IRSs, condition (disintegration) to arrive at probability measures that can be understood.

$A =$





$$H^{A}_{u,v} = \left\{ H : H \text{ contains an element } \gamma \text{ that moves } T_u \text{ to } T_v, \text{ and } \gamma \in \mathcal{O}_{A,0} \right\}$$

Expand n steps down,

the projection of $H \cap O_{A,n}$ to $\mathrm{Sym}(\partial A_n)$ cannot be small.