Asymptotic Expander Graphs

Federico Vigolo

Symmetries (Newcastle 17/8/20)

Plan of the talk:

- ★ Definition of expanders
- ★ Spectral characterization & a construction
- ★ Expanders and Baum-Connes conjectures (sketchy)
- ★ Quasi-local projections (sketchy)
- ★ Asymptotic expander graphs
- ★ Exhaustions by expanders & consequences
- ★ Further comments (time permitting)

this is for motivation }

Slogan Definition:

"Expander graphs are sequences of sparse yet highly connected finite graphs"

we need arbitrarily large

uniformly bounded degree

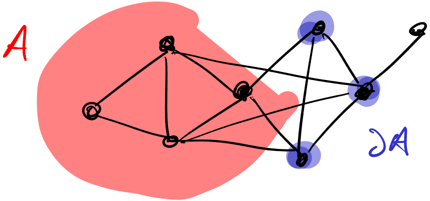
to disconnect them it is necessary to remove many edges ⟺ it is easy to walk from one vertex to any other

Some applications:

- ★ computer science (error correcting codes, random algorithms, ...)
- ★ coarse embedding into Hilbert spaces (Gromov monsters, Baum-Connes conjectures)

Actual definition:

$\mathcal{G}$ finite unoriented simplicial graph



Identify $\mathcal{G}$ with its set of vertices $\qquad \deg(\mathcal{G}) = \max\limits_{v \text{ vertex}} \deg(v)$

$A \subset \mathcal{G} \rightsquigarrow$ s.set of vertices $\qquad |\mathcal{G}| = \#$ vertices

Def Given $\varepsilon > 0$, $\mathcal{G}$ is an ε-expander if $\forall A \subset \mathcal{G}$, $|A| \leq \frac{1}{2}$

$$|\partial A| > \varepsilon |A|$$

this is an isoperimetric inequality

$$\partial A := \{ w \in \mathcal{G} \setminus A \mid \exists v \in A \quad v \overset{\text{edge}}{\sim} w \}$$

$=$ vertices at distance 1 from A

Rmk this means that $\mathcal{G}$ is hard to disconnect

Not a tree!

← to cut $\mathcal{G}$ in half need to remove $\approx \varepsilon |\mathcal{G}|$ edges

Def a family of expanders is a sequence of finite graphs $(G_n)_{n\in\mathbb{N}}$ s.t.

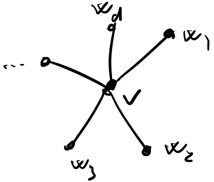
- ★ $|G_n| \longrightarrow \infty$
- ★ $\deg(G_n) \leq d \quad \forall n$
- ★ G_n is ε-expander $\forall n$

Thm (Pinsker) Let $d \geq 3$. A random sequence $(G_n)_{n\in\mathbb{N}}$ of connected d-regular graphs with $|G_n| \longrightarrow \infty$ is almost surely a family of expanders

We need concrete examples

Consider the averaging operator: $A: \mathbb{R}^{\mathcal{G}} \longrightarrow \mathbb{R}^{\mathcal{G}}$

$$Af(v) = \frac{1}{\deg(v)} \sum_{w \sim v} f(w)$$



Rmk if $\mathcal{G}$ is d-regular then $A = \frac{1}{d}$(adjacency matrix) is a symmetric matrix
$\Longrightarrow$ it has real eigenvalues $1 = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{|\mathcal{G}|}$

Th (Alon-Millman, Dodziuk) A sequence of d-regular graphs $\mathcal{G}_n$
with $|\mathcal{G}_n| \longrightarrow \infty$ is a family of expanders i.f.f.
$\exists\, \delta > 0$ s.t. $\lambda_2 \leq 1 - \delta$ for every $\mathcal{G}_n$

this is a formalization of "it is easy to walk around" in the sense that the
Rmk Random Walk on $\mathcal{G}_n$ converge uniformly exponentially fast to the constant
functions

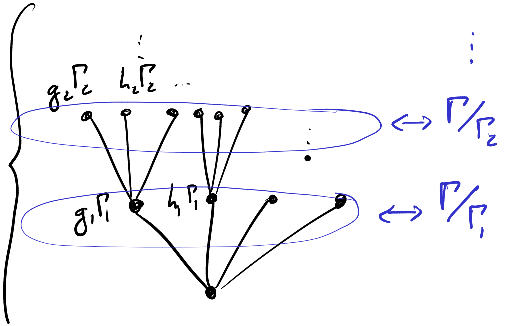
Construction

Let $\Gamma = \langle S \rangle$ be a finitely generated group,

$\Gamma > \Gamma_1 > \Gamma_2 > \dots$ a chain of finite index subgroups

$\rightsquigarrow$ construct a coset tree T

$T =$



Γ acts on the boundary of the tree $\partial_\infty T$ preserving the uniform probability measure

Easy consequence of Alon-Millman, Dodziuk:

If the action $\Gamma \curvearrowright \partial_\infty T$ has spectral gap, then $Sch(\Gamma/\Gamma_n, S)$ is a family of expanders

$\Gamma \curvearrowright L^2(\partial_\infty T)$ by precomposition, $A_S : L^2(\partial_\infty T) \longrightarrow L^2(\partial_\infty T)$ self adjoint.

$$A_S f = \frac{1}{|S|} \sum_{s \in S} s \cdot f$$

$\rightsquigarrow$ spectral gap if $\lambda_2 < 1$

Eg if Γ has Kazhdan's prop (T) then any chain works

Less easy (but not hard): the converse is also true

The Baum-Connes conjectures (BCC) predict that some assembly maps are isomorphisms of K-theories

↑ there are various generalizations → BCC with coefficient
→ coarse BCC

Thm [Yu] If a (bounded geometry, discrete) metric space admits a coarse embedding into an Hilbert space then it satisfies the coarse BCC.

each finite graph embed, but the metric distortion increases
↑
Easy fact: expanders do not uniformly coarsely embed into Hilbert spaces.

Thm [Gromov, Osajda] There exist finitely generated groups whose Cayley graphs (coarsely) contain families of expanders (Gromov's monsters)

Thm [Higson, Higson-Lafforgue-Skandalis] Expanders can be used to construct counter examples to the coarse BCC

Key point: consider the mean value operator $P_{G_n} : \mathbb{R}^{G_n} \longrightarrow \{\text{const. functions}\}$

$$f \longmapsto \frac{1}{|G_n|} \sum_{v \in G_n} f(v)$$

the 2 is to get rid of negative eigenvalues ↘

If G_n is an ε-expander then $A^{2k} \longrightarrow P_{G_n}$ exponentially fast (with exponent depending on ε) because the constant functions are the 1-eigenspace and all other eigenvalues are $\leq 1-\delta$ (this is just a restatement that the Random Walks converge quickly)

Easy consequence $\rightsquigarrow$ if G_n is a sequence of expanders, then the operator

$$P := \bigoplus_{\mathbb{N}} P_{G_n} : \bigoplus_{\mathbb{N}} \mathbb{R}^{G_n} \longrightarrow \bigoplus_{\mathbb{N}} \mathbb{R}$$

belongs to the Roe algebra of $\bigsqcup G_n$ (and it is a non-compact ghost projection)

More work $\rightsquigarrow$ The coarse assembly map is not surjective $\Rightarrow$ coarse BCC is false

More precisely, the Roe algebra $C^*(\sqcup G_n)$ is a sub-algebra of the C^*-algebra of bounded operators $\mathcal{B}(\mathcal{H})$ where $\mathcal{H} = \bigoplus_{n\in\mathbb{N}} \mathbb{R}^{G_n}$

the image of the assembly map is in it's K-theory. ↖ Hilbert sum

$P = \bigoplus_{n\in\mathbb{N}} P_{G_n} \in \mathcal{B}(\mathcal{H})$ is the orthogonal projection to the space of functions that are constant on G_n $\forall n \in \mathbb{N}$. $A^{2k} \longrightarrow P \Rightarrow P \in C^*(\sqcup G_n)$

Easy classical fact: every element of the Roe algebra is a "quasi-local operator"

Open Question: is the converse true?

Sub-question: how about P?

Easy fact: for a sequence of finite graphs $(G_n)_{n\in\mathbb{N}}$, the projection P is quasi-local $\Longleftrightarrow$ $\Big[$ $\forall \varepsilon>0 \quad \exists R>0$ s.t. if $A,B \subseteq G_n$ are s.t. $d(A,B)>R$ then either $|A| \leq \varepsilon |G_n|$
$|B| \leq \varepsilon |G_n|$

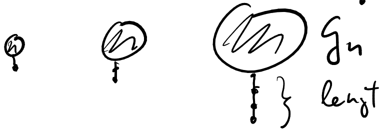
we can take this as definition of q. loc.

Easy exercise: check directly that if G_n is a sequence of expanders then P is quasi-local

Warm-up Question: G_n bded degree graphs + P quasi-local $\overset{?}{\Longrightarrow}$ $(G_n)_{n\in\mathbb{N}}$ family of exp. ?

No.

Example: let G_n be a sequence of expanders and set $G_n' = G_n \cup \{\text{a small tail}\}$



Then P is quasi-local (very easy)

Thm [Li-Nowak-Špakula-Zhang] Suppose that $\deg(G_n) \leq d$, then P is quasi-local iff

$$\forall\, 0 < \alpha < \tfrac{1}{2} \quad \exists\, \varepsilon > 0 \quad \text{s.t.} \quad |\partial A| \geq \varepsilon |A| \quad \forall A \subset G_n \quad \boxed{\alpha |G_n|} \leq |A| \leq \tfrac{1}{2}|G_n|$$

this is the only difference from usual expanders

Def a family of such graphs is a sequence of asymptotic expanders (i.e. as. exp $\Longleftrightarrow$ P is q.loc)

Rmk the actual definition (and the theorem) are more general, as it is concerned with sequences of finite metric spaces (it requires adding an extra parameter to define $\partial_R A$)

More interesting example $\Gamma > \Gamma_1 > \Gamma_2 > \dots$ chain of finite index sgps

then $Sch(\Gamma/\Gamma_n, S)$ is a seq of as. exp iff.

$\Gamma \curvearrowright \partial_\infty T$ is strongly ergodic

if $A_k \subset \partial_\infty T$ is a seq. of measurable ssets such that $\mu(A_k \Delta \gamma(A_k)) \xrightarrow{k\to\infty} 0 \ \forall \gamma \in \Gamma$ then $\mu(A_k)(1-\mu(A_k)) \longrightarrow 0$

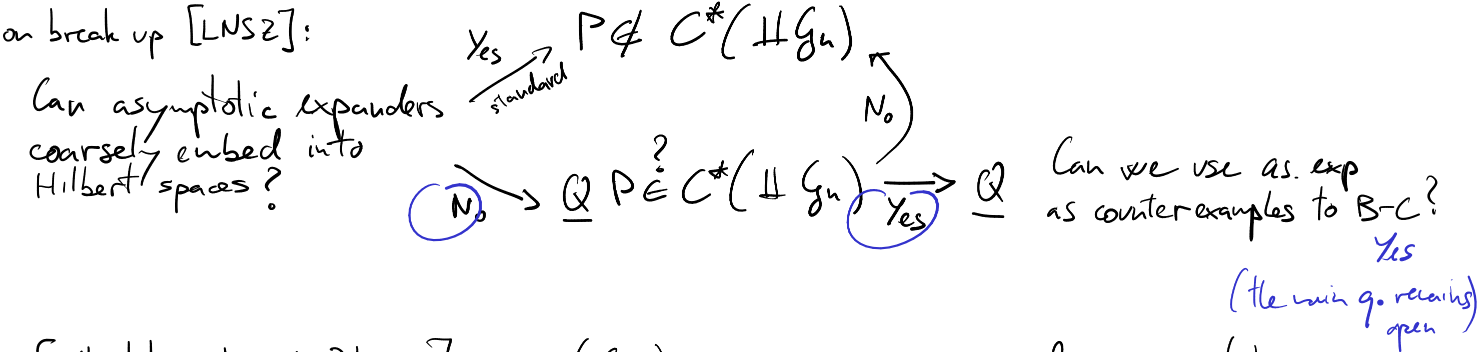
Thm [Abért-Elek] 1) if $\Gamma_k \lhd \Gamma$ $\forall k$, then strong. ergodic $\Longleftrightarrow$ Spectral gap

2) $\exists$ a (non normal) chain s.t. $\Gamma \curvearrowright \partial_\infty T$ is strong. erg but has no spec. gap (probabilistic argument)

Rmk It would be interesting to have an explicit construction

Back to our question: is it true that $P \in C^*(\amalg G_n) \Longleftrightarrow P$ quasi-local ?

Question break up [LNS2]:



Can asymptotic expanders coarsely embed into Hilbert spaces?

$P \notin C^*(\amalg G_n)$

Q $P \overset{?}{\in} C^*(\amalg G_n)$

Q Can we use as. exp as counterexamples to B-C?

Yes

(the main q. remains open)

Thm [Khukhro-Li-V-Zhang] $(G_n)_{n\in\mathbb{N}}$ is a sequence of asymptotic expanders iff it admits an exhaustion by expanders

$\Big($ $\exists$ $G_n^0 \subseteq G_n^1 \subseteq G_n^2 \subseteq \dots$ s.t. $G_n^k = G_n$ for k large enough
and $\forall k$ $(G_n^k)_{n\in\mathbb{N}}$ is a seq of expanders $\Big)$

depends on n

Key lemma (Maximality Trick) Fix $\varepsilon > 0$ and let $A \subset G$ be a maximal subset s.t. $|A| \leq \frac{1}{2}|G|$ and $|\partial A| \leq \varepsilon |A|$.

Then $\forall\ B \subseteq G \setminus A$ s.t. $|A| + |B| \leq \frac{1}{2}|G|$

$$|\partial B \cap (G \setminus A)| > \varepsilon |B|$$

← i.e. the subgraph $G \setminus A$ wants to be an expander.

Sketch of the proof of thm

G_n as. expander → large subsets have linearly large ∂

→ for ε small enough, $\exists$ small maximal ε-Følner set A

⇓

small subsets of $G_n \setminus A$ have lin. large ∂

→ profit

the converse is easy

An aside:

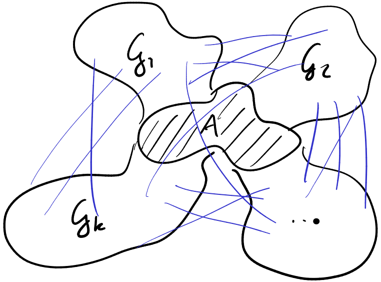
Rmk for general graphs, the key lemma can be used to prove the following:

Thm [V] if $\mathcal{G}$ is any connected graph s.t. $\exists\, A \subset \mathcal{G}$ maximal ε-Følner set with $|A| \leq (\frac{1}{2} - \delta)|\mathcal{G}|$, then

$$\mathcal{G} \setminus A = \mathcal{G}_1 \cup \dots \cup \mathcal{G}_k$$

where each $\mathcal{G}_i$ is ε-expander and $|\mathcal{G}_i| \geq \delta |\mathcal{G}|$

$\mathcal{G} =$



Back to asymptotic expanders:

<u>Comment 1</u>: Recall the example of profinite actions: $\Gamma > \Gamma_1 > \Gamma_2 > \cdots$

$$\begin{array}{lcccl} \text{Then } Sch(\Gamma/\Gamma_i, S) & \text{expanders} & \iff & \Gamma \curvearrowright \partial_\infty T & \text{has spectral gap} \\ \quad \cup & & & \cup & \\ & \text{as. exp.} & \iff & & \text{is str. ergodic} \end{array}$$

This is a special case of a general phenomenon relating graphs & actions

$$\begin{tikzcd} \text{Graphs} \arrow[r, shift left, "\text{realize as Schreier}"] & \text{Actions on measure spaces} \arrow[l, shift left, "\text{finite approximation}"] \end{tikzcd}$$

expanders $\longleftrightarrow$ expansion in measure $\iff$ spectral gap

as. expanders $\longleftrightarrow$ asymp. expansion in measure $\iff$ strong. ergodicity

Comment 2

Recall [Abért-Elek]: $\Gamma_n \lhd \Gamma \Longrightarrow (\Gamma \curvearrowright \partial_\infty T$ strong. erg $\Longleftrightarrow$ spec. gap$)$

happens iff $\mathrm{Sch}(\Gamma/\Gamma_n) = \mathrm{Cay}(\Gamma/\Gamma_n)$ is transitive

[K-L-V-Z] The same holds for general graphs:

$(G_n)_{n\in\mathbb{N}}$ vertex-transitive $\Longrightarrow$ (as. expanders $\Longleftrightarrow$ expanders)

More in general, the same holds for actions on measure spaces:

$\Gamma \curvearrowright (X,\mu)$ p.m.p commuting with an ergodic action

dynamical analogue of transitivity

$\Longrightarrow$ (strong erg $\Longleftrightarrow$ spec. gap)

[Chifan-Ioana, L-V-Z]

Thank You !