REWRITING SYSTEMS AND GEODETIC GRAPHS

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- geodetic graphs (graph theory)
- plain groups (group theory)
- finite convergent length-reducing rewriting systems (computer science)
- what do any of these things have to do with any other?

Let Δ be a simple (no loops or multiple edges) undirected graph.

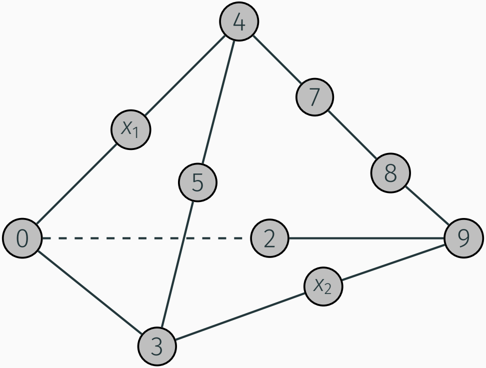
A path of length n in Δ is a sequence of vertices $u_0, u_1, \ldots, u_n$ with the property that u_i and u_{i+1} are adjacent.

Each connected graph Δ is equipped with a natural metric in which the distance between two vertices is the length of the shortest path between them.

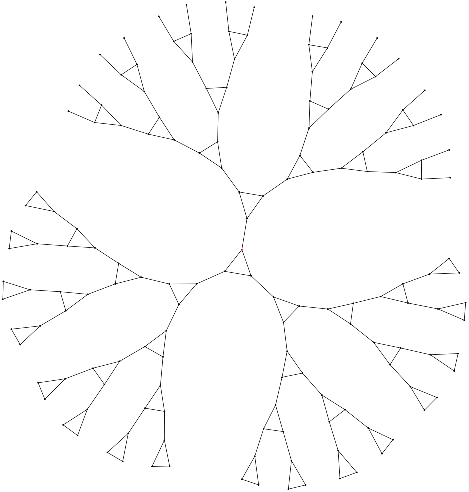
The shortest path between any two vertices is called a *geodesic*.

Δ is *geodetic* if the geodesic between any two vertices is unique.

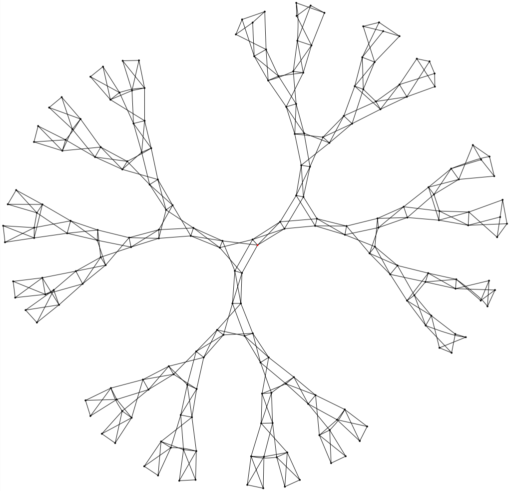
EXAMPLE



EXAMPLE



EXAMPLE



A path in Δ is an *embedded circuit* (EC) if the vertices $u_0, \ldots, u_{n-1}$ are distinct and $u_0 = u_n$.

An embedded circuit in Δ is *isometrically embedded* (IEC) if the subgraph comprising the vertices in the circuit and the edges between consecutive vertices is convex in Δ;

that is, $d(u_i, u_j) = \min\{j - i, n + i - j\}$ for all $0 \leq i < j < n$.

Quiz 1: if Δ is geodetic, length of an IEC must be ... odd.

Cool fact (in Andrew Elvey Price's MSc thesis)

Lemma

Let Δ be a geodetic graph, and let $u_0, u_1, \ldots, u_n$ and $u_0, u_1', \ldots, u_n'$ be equal length geodesics in Δ such that $u_1 \neq u_1'$ and $d(u_n, u_n') = 1$. Then

$$u_0, u_1, \ldots, u_n, u_n', \ldots, u_1', u_0$$

is an IEC.

Ore (1962, “Theory of graphs”): classify them!

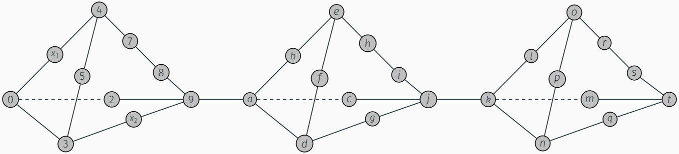
Build them: method to construct new ones: eg start with K_n and . . .

Shapiro (1997): What groups admit geodetic Cayley graphs?

A vertex v in Δ is a *cut vertex* if Δ is connected, but the graph obtained from Δ by removing v and the edges incident to v is disconnected.

A graph is *two-connected* if it is connected and has no cut vertices.

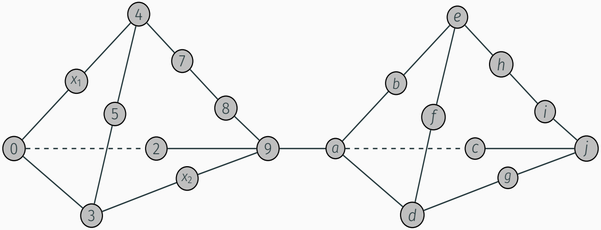
The maximal two-connected subgraphs of a graph Γ are called *blocks*.



It follows immediately from the maximality of blocks that any block B in Δ is the subgraph of Δ induced by the vertex set of B.

Lemma (Blocks vs. ECs)

Let Δ be a simple undirected graph. Two vertices u, v of Δ lie in the same block if and only if there exists an embedded circuit in Δ that visits both.



.

Given a connected graph Δ, the *block-cut tree* $T = T(\Delta)$ is a well-known construction which encodes the block structure of Δ.

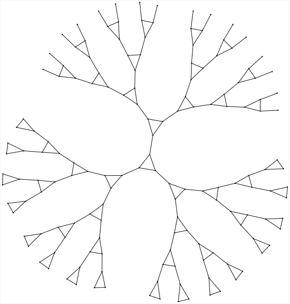
– one vertex v_x (of type I) for each vertex x of Δ, and one vertex v_B (of type II) for each block B of Δ;

– a type I vertex v_x is adjacent in T to a type II vertex v_B if x is a vertex in the block B.

For any connected graph Δ, the block-cut tree $T(\Delta)$ is a tree (every embedded circuit has length at most two).

Definition

A group is *plain* if it isomorphic to a free product of finitely many factors, with each factor a finite group or an infinite cyclic group.



Bass-Serre Theory tells us that a group G is plain if and only if G acts geometrically on a locally-finite tree, with finite vertex stabilisers and trivial edges stabilisers.

Theorem (Plain groups and ECs)

For a group G and a positive integer s, the following are equivalent:

1. *G admits a finite generating set Σ such that, in the associated undirected Cayley graph $\Gamma(G, \Sigma)$, the diameter of any embedded circuit is at most s.*
2. *G admits a finite generating set Σ such that, in the associated undirected Cayley graph $\Gamma(G, \Sigma)$, the diameter of any block is at most s.*
3. *G is a plain group.*

1. $\Leftrightarrow$ 2.: follows immediately from the Lemma above (Blocks vs. ECs).

Any group G acts on the block-cut tree of its (undirected) Cayley graph.

If this graph has finite diameter blocks (and its locally-finite), then G is acting on a tree with finite vertex stabilisers and (because of the block-cut construction) *trivial* edges stabilisers.

So Bass-Serre Theory tells us G is plain.

A *rewriting system* is a pair (Σ, T) that formalises the idea of working with products from a set of allowable symbols, using a set of simplifying rules.

Σ is a nonempty set, called an *alphabet* of *letters*. T is a possibly empty subset of $\Sigma^* \times \Sigma^*$, called a set of *rewriting rules*.

The set of rewriting rules determines a relation $\to$ (immediately reduces to) on the set Σ^* by the following rule: $a \to b$ if $a = u\ell v$, $b = urv$ and $(\ell, r) \in T$.

The reflexive and transitive closure of $\to$ is denoted $\xrightarrow{*}$ (reduces to).

A word $u \in \Sigma^*$ is *irreducible* if no factor is the left-hand side of any rewriting rule, and hence $u \xrightarrow{*} v$ implies that $u = v$.

The reflexive, transitive and symmetric closure of $\rightarrow$ is called "equivalence", and denoted $\overset{*}{\leftrightarrow}$.

The operation of concatenation of representatives is well defined on the set of $\overset{*}{\leftrightarrow}$-equivalence classes, and hence makes a monoid $M = M(\Sigma, T)$.

We say that M is the monoid presented by (Σ, T). When the equivalence class of every letter has an inverse, the monoid M is a group and we say it is *the group presented by* (Σ, T).

Quiz 2: Let $\Sigma = \{a, A\}$ and let $T = \{(aA, 1), (Aa, 1)\}$.

Then (Σ, T) presents $\mathbb{Z}$.

Quiz 3: Let G be a finite group. Let $\Sigma = G \setminus \{e\}$ and let $T =$

$\left\{(gh, k) \mid g, h, k \in \Sigma \text{ and } gh =_G k\right\} \cup \left\{(gh, 1) \mid g, h \in \Sigma \text{ and } g =_G h^{-1}\right\}.$

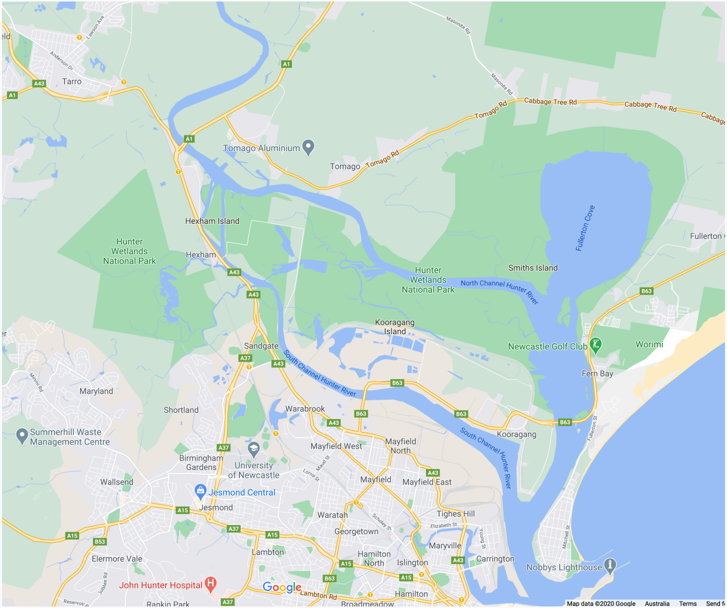
Then (Σ, T) presents G.

A rewriting system (Σ, T) is

- *finite* if Σ and T are finite sets,
- *terminating* (or *noetherian)* if there are no infinite sequences of allowable factor replacements,
- *length-reducing* if for all $(\ell, r) \in T$ we have that $|\ell| > |r|$.

Two words x and y are called *joinable* if there exists $z \in \Sigma^*$ such that x and y both reduce to z.

A rewriting system is called *confluent* if whenever $w \xrightarrow{*} x$ and $w \xrightarrow{*} y$, then x and y are joinable.


Tarro
Tomago Aluminium
Tomago
Tomago Rd
Cabbage Tree Rd
Hexham Island
Hexham
Hunter Wetlands National Park
Hunter Wetlands National Park
North Channel Hunter River
Smiths Island
Fullerton Cove
Kooragang Island
Newcastle Golf Club
Worimi
Fern Bay
Sandgate
South Channel Hunter River
Maryland
Shortland
Warabrook
Kooragang
South Channel Hunter River
Summerhill Waste Management Centre
Mayfield West
Mayfield North
Birmingham Gardens
University of Newcastle
Wallsend
Mayfield
Mayfield East
Jesmond Central
Jesmond
Waratah
Tighes Hill
Georgetown
Maryville
Elermore Vale
Lambton
Hamilton North
Islington
Carrington
Nobbys Lighthouse
John Hunter Hospital
Rankin Park
Lambton Rd
Broadmeadow
Hamilton
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Terms

A rewriting system is called *convergent* if it is terminating and confluent.

It follows that in a convergent [length-reducing] rewriting system, rewriting any word in Σ^* until you can rewrite no more

is an algorithm for producing the unique irreducible [geodesic] word representing the same element.

Lemma (FCLRRS implies geodetic)

If G is presented by a FCLRRS (Σ, T) with $\Sigma = \Sigma^{-1}$,

then its undirected Cayley graph (wrt Σ) Γ must be geodetic.

Moreover, if the left sides of rules in T have length at most $s+1$,

then IECs in Γ have length at most $2s+1$.

Lemma (Combining rewriting system to present free products)

Suppose that $(\Sigma_1, T_1), \ldots, (\Sigma_n, T_n)$ *are rewriting systems presenting groups* $G_1, \ldots, G_n$ *respectively and such that the alphabets* $\Sigma_1, \ldots, \Sigma_n$ *are pairwise disjoint.*

The combined rewriting system $\left(\cup_{i=1}^{n}\Sigma_i, \cup_{i=1}^{n}T_i\right)$ *presents the free product* $G_1 * \cdots * G_n$.

Corollary

If G *is a plain group, then* G *admits presentation by a finite convergent length-reducing rewriting system* (Σ, T) *where* $\Sigma = \Sigma^{-1}$ *and the left-hand side of every rule has length equal to two.*

(Madlener and Otto) Prove a complete algebraic characterisation of groups presented by length-reducing systems.

Conjecture (Gilman (1984))

Let G be a group. Then G admits presentation by a finite convergent length-reducing rewriting system (Σ, T) in which the right-hand side of every rule has length at most one if and only if G is plain.

Conjecture (Madlener and Otto (1987))

Let G be a group. Then G admits presentation by a finite convergent length-reducing rewriting system (Σ, T) if and only if G is plain.

Eisenberg and Piggott (2019): proved Gilman's Conjecture

and then

Theorem 1 (E, Piggott (2020))

Let G be a group. Then G admits presentation by a finite convergent length-reducing rewriting system (Σ, T) *such that* $\Sigma = \Sigma^{-1}$ *and the left-hand side of every rule has length at most three if and only if G is plain.*

Our proof is via geodetic graph theory.

The backwards direction is done: if G is plain, you can explicitly construct a rewriting system (choosing Σ to be every non-trivial element of every finite factor, plus one generator and its inverse for each $\mathbb{Z}$ factor. Corollary above).

The forwards direction: Lemma (FCLRRS implies geodetic) gives us the undirected Cayley graph is geodetic and IECs length at most 5.

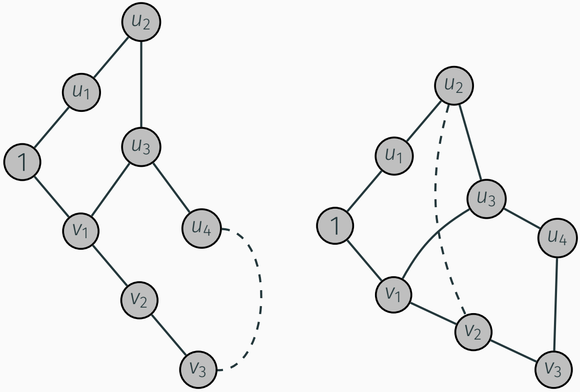
If we could connect this up to say something about the diameter of *embedded* circuits (not nec IEC), then we have Theorem (Plain groups and ECs — the block-cut tree stuff) which gives plain.

Theorem 2 (E, Piggott)

If Γ is a locally-finite geodetic undirected simple graph in which isometrically embedded circuits have length at most 5, then all embedded circuits have diameter at most 2.

Proof strategy: assume you have a minimial length counterexample. Argue a lot. Then get a contradiction.

- if ρ is a minimal length counterexample (an EC of diameter at least 3) then ρ contains a geodesic subpath of length 3.
- if ρ is a minimal counterexample in a geodetic graph with IECs length at most 5, then ρ must look like this:



FINAL WORDS

While Theorem 1 (rewriting left sides length ≤ 3 iff plain) falls well short of resolving Madlener and Otto's conjecture,

and Theorem 2 (IECs length ≤ 5 implies ECs diameter ≤ 2) is an incremental contribution to our understanding of geodetic graphs,

we think our proof shows how far you can get (or maybe one can push it further) using a primarily graph-theoretic approach.

THANKS