Kaplansky's conjectures

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Group rings

Let G be a group and R be a ring. The group ring $R[G]$ is the ring

$$\left\{ \sum r_g g \mid r_g \in R, g \in G \right\}$$

of finite formal sums with multiplication

$$\left(\sum r_g g \right) \cdot \left(\sum s_h h \right) := \sum (r_g s_h)(gh).$$

Note that $g \mapsto 1_R g$ is an embedding $G \hookrightarrow (R[G])^\times$ of G in the group of units and $r \mapsto r 1_G$ embeds R as a subring of $R[G]$. An expression like $r - g$ makes sense.

G-actions on R-modules (e.g. vector spaces) are the same thing as $R[G]$-modules, so group rings are natural objects of study in representation theory, topology, etc. If K is a field then $K[G]$ is an algebra over K and one might say "group algebra" instead of "group ring".

Group rings (continued)

If $G = \mathbb{Z} = \langle t \rangle$ then $R[G]$ is just finite formal sums of multiples of powers of t, that is, the Laurent polynomials $R[t, t^{-1}] = \sum_i a_i t^i$.

Similarly $R[\mathbb{Z}^k]$ is just the Laurent polynomials in k unknowns.

Gromov's dichotomy

When we are dealing with statements for all groups (or all countable groups, or all torsion-free groups, or...) we must not forget:

Gromov's dichotomy

Any statement for all countable groups is either trivial or false.

However, we have to take it with a grain of salt as well:

- The group ring of a torsion-free group over a field K is prime: if $arb = 0$ for all $r \in K[G]$ then $a = 0$ or $b = 0$ (Connell, 1963).
- Every countable group embeds into an almost finitely presented group (Leary, 2018).

Zero divisors

Understanding zero divisors is a basic step in studying a ring. For instance, they obstruct embedding into a skew field (a.k.a. division ring).

Suppose that $g \in G$ has finite order $n \geq 2$. Then $(1-g)(1+g+\cdots+g^{n-1}) = 1-g^n = 0$ so $R[G]$ has a non-trivial zero divisor.

From this point on, we will only consider torsion-free groups.

Note that $R[\mathbb{Z}]$ has no zero-divisors.

Kaplansky's zero divisor conjecture

Let G be a torsion-free group and let K be a field. Then the group ring $K[G]$ has no non-trivial zero divisors.

Since $\mathbb{Z}[G] \subset \mathbb{C}[G]$, working over a field instead of a ring still covers arguably the most basic case of a group ring.

Zero divisors (continued)

The problem is old! Kaplansky included it in a 1956 problem list.

> **On the other hand, if G has no such element x, that is, if G is torsion free, then $K[G]$ has at least no obvious divisors of zero. Because of this, and with frankly very little supporting evidence, it was conjectured that if G is torsion free, then $K[G]$ has no zero divisors. Amazingly, this conjecture has held up for over 30 years.**

Figure 1: Excerpt from the 1977 edition of Passman's book *The Algebraic Structure of Group Rings*

There are 2 other conjectures ruling out certain types of elements. Note that if $r \in R^\times$ is a unit of the base ring and $g \in G$ is any group element, then rg is a unit with inverse $r^{-1}g^{-1}$; such units are called *trivial units*.

Recall that an *idempotent* is simply a solution to $x^2 = x$.

Let G be a torsion-free group and let K be a field.

- unit conjecture: $K[G]$ has no non-trivial units.
- zero divisor conjecture: $K[G]$ has no non-trivial zero divisors.
- idempotent conjecture: $K[G]$ has no idempotents other than 0 and 1.

These are all still open.

Relationship between the conjectures

unit conjecture $\implies$ zero divisor conjecture $\implies$ idempotent conjecture

A non-trivial idempotent x is a zero divisor since $x(x-1) = x^2 - x = 0$.

Turning a zero divisor into a non-trivial unit is a little more work. Suppose that $ab = 0$ for some non-zero $a, b \in K[G]$. Since $K[G]$ is prime we can find $c \in K[G]$ such that $bca \neq 0$. Now $(bca)^2 = bc(ab)ca = 0$ so that $(1 + bca)(1 - bca) = 1$ and we have non-trivial units (after quickly thinking about the minor technicality in characteristic 2).

If the zero divisor conjecture holds, every left-invertible element is a unit: $ab = 1$ implies that $a(ba - 1) = aba - a = 0$ and hence $ba = 1$.

What is known

Kaplansky's conjectures follow from other bold conjectures for torsion-free groups that are subject to Gromov scepticism (or were for a few decades).

- Unique products $\implies$ unit conjecture
- Atiyah conjecture $\implies$ zero divisor conjecture over $\mathbb{C}$
- Baum–Connes conjecture or Farrell–Jones conjecture $\implies$ idempotent conjecture over $\mathbb{C}$

A group G *has unique products* if for finite subsets $A, B \subset G$ there is always some element uniquely expressible as ab for $a \in A, b \in B$.

Residually torsion-free solvable groups satisfy Atiyah.

Hyperbolic groups and amenable groups satisfy Baum–Connes and hyperbolic groups and CAT(0) groups satisfy Farrell–Jones.

The zero divisor conjecture actually holds for elementary amenable groups over any field.

What is known (continued)

Lemma (Botto Mura–Rhemtulla, 1975)

Left-orderability implies unique products.

Left-orderable groups include locally indicable groups (e.g. torsion-free nilpotent groups and one-relator groups, free-by-cyclic groups, special groups, Thompson's group F), recovering Higman's 1940 theorem.

Proof.

Enumerate the finite subsets $A, B \subset G$ as $a_1 < a_2 < \cdots < a_m$ and $b_1 < b_2 < \cdots < b_n$. Then $a_i b_1 < a_i b_j$ for all $j \neq 1$ and thus the minimal product $a_i b_j$ must have $j = 1$. The m possibilities $a_1 b_1, \ldots a_m b_1$ are distinct group elements, so the minimal product is unique. (NB: it will *not* be $a_1 b_1$ in general, because we only assume the order to be left-invariant!) □

It turns out that finding groups without unique products is hard. Maybe unique products is actually equivalent to the unit conjecture?

Non-unique product groups

The concept of unique product groups was introduced by Rudin and Schneider in 1964.

The first example of a torsion-free group without unique products was constructed by Rips and Segev in 1988 using small cancellation. Shortly thereafter, Promislow showed that the group

$$P = \langle\, a, b \mid (a^2)^b = a^{-2},\, (b^2)^a = b^{-2} \,\rangle$$

which is an extension of $\mathbb{Z}^3$ by $\mathbb{Z}/2 \times \mathbb{Z}/2$ contains a 14-element set A such that $A \cdot A$ has no unique product.

The group P is known variously as the Hantzsche–Wendt group, Passman group, Promislow group, and Fibonacci group $F(2,6)$, and is the unique 3-dimensional crystallographic group with finite abelianization.

Non-unique product groups (continued)

The known groups without unique products come in two flavours:

- small cancellation: Rips–Segev (1988), Steenbock (2015), Gruber–Martin–Steenbock (2015), Arzhantseva–Steenbock (2014+)
- small presentation: Promislow (1988), Carter (2014), Soelberg (2018)

All examples (thus far) of the second flavour are known to satisfy the zero divisor conjecture.

There has been serious effort dedicated to understanding what potential non-trivial units for the group ring of Promislow's group could be (Craven–Pappas 2013, Abdollahi–Zanjanian 2019). If the converse "unit conjecture $\implies$ unique products" is true, it cannot be true "locally".

> *To date it has never been clear whether the unit conjecture is a ring-theoretic problem. We hope that this paper suggests that it might be.* Craven–Pappas

What's new

What is my contribution to the Kaplansky story?

Non-trivial units

The unit conjecture is false.

Theorem (G., 2021)

Let P be the torsion-free, virtually abelian group

$$\langle\, a, b \mid (a^2)^b = a^{-2}, (b^2)^a = b^{-2} \,\rangle.$$

There is a non-trivial unit $\alpha \in \mathbb{F}_2 P$ such that $|\operatorname{supp} \alpha| = \left|\operatorname{supp} \alpha^{-1}\right| = 21$.

Questions?