Irreducible Pythagorean Representations of R. Thompson's Groups

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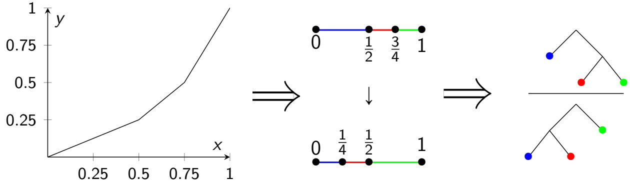
Symmetry in Newcastle

August 2023

Richard Thompson's Groups $F \subset T \subset V$ [Thompson 65, Brown 87]

F is the group of piece-wise, linear homeomorphisms of $[0, 1]$ with finitely many non-differentiable points which are contained in the dyadic rationals.



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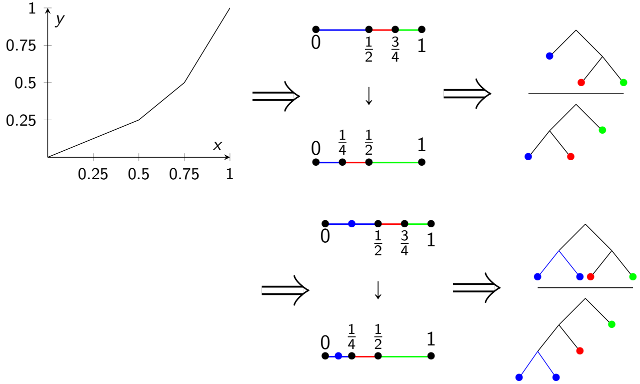

Exceptional Properties

1. T, V first examples of infinite but finitely presented, simple groups [Thompson 65]
2. F is first example of torsion-free, infinite-dimensional group of type F_∞ [Brown-Geoghegan 84]

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Known Irreducible Representations of F

1. Rep induced from the Cuntz algebra [Birget 04, Nekrashevich 04, Barata-Pinto 19, Arujo-Pinto 20, Guimaraes-Pinto 22];
2. Bernoulli reps for $0 < p < 1$ and $\phi \in S_1$ [Garncarek 12, Olesen 16] ;
3. Jones' rep coming from certain trivalent tensor categories [Jones 19].

Vaughan Jones' Machinery [Jones-17]

Jones' machinery: from simple objects build complicated objects.

- Jones' machinery can used to build representations of F:
  - **Simple objects:** (Finite-dimensional) Hilbert space $\mathfrak{H}$ and an isometry between Hilbert spaces.
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Theorem (Important applications)

1. *New proof of $[F, F], T, V$ are not Kazhdan groups [Brothier-Jones 19];*
2. *First known example of reps of F that are Ind-mixing [Brothier-W 22].*
3. *New families of irreducible reps of F [Jones 19], [Brothier-W 23];*

Pythagorean Module

Definition

A *Pythagorean module* (P-module) is a triple $(A, B, \mathfrak{H})$ where $\mathfrak{H}$ is a Hilbert space, $A, B \in B(\mathfrak{H})$ satisfying the *Pythagorean relation*

$$A^*A + B^*B = \mathrm{id}.$$

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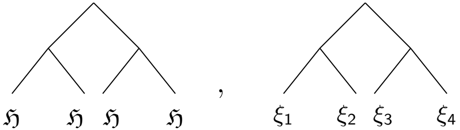
- An *intertwinner* between P-modules $(A, B, \mathfrak{H})$, $(A', B', \mathfrak{H}')$ is a bounded linear map $\theta : \mathfrak{H} \to \mathfrak{H}'$ satisfying

$$\theta \circ A = A' \circ \theta \text{ and } \theta \circ B = B' \circ \theta.$$

- P-modules are (unitarily) *equivalent* if there exists a unitary intertwinner between them.
- A *sub-module* of $(A, B, \mathfrak{H})$ is a Hilbert subspace $\mathfrak{H}' \subset \mathfrak{H}$ that is closed under A, B.

- Consider a P-module $(A, B, \mathfrak{H})$.
- Associate each tree t with the direct sum $\mathfrak{H}_t := \mathfrak{H}^{\mathrm{Leaves}(t)}$.

$\mathfrak{H}_t =$

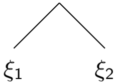


$\in \mathfrak{H}_t$

- Let $\mathscr{H} = \mathscr{H}_{A,B}$ be the completion of $\bigsqcup_{t \text{ is a tree}} \mathfrak{H}_t / \cong$.
- $\mathscr{H}$ is always an *infinite-dimensional* Hilbert space.

Jones' Machinery: Pythagorean Reps [Jones-17, Brothier-Jones 19]

- Representatives of $[\wedge, (\xi_1, \xi_2)]$:



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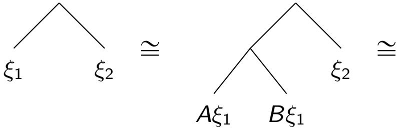

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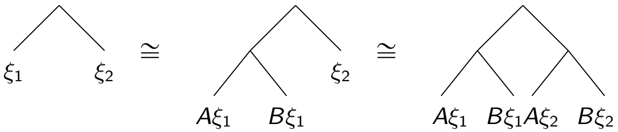

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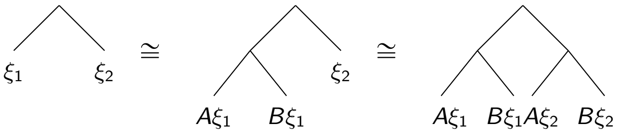

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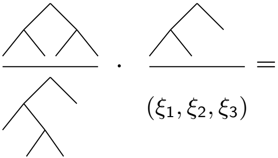

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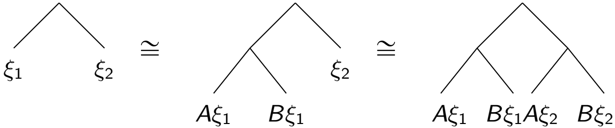

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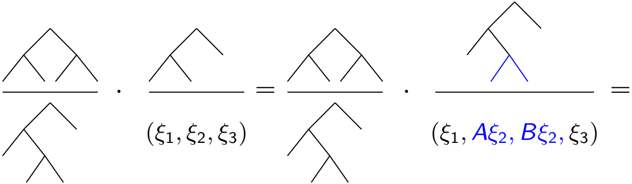

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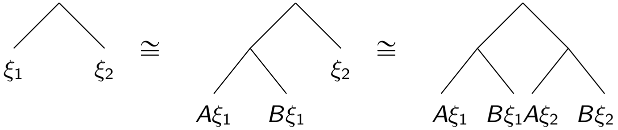

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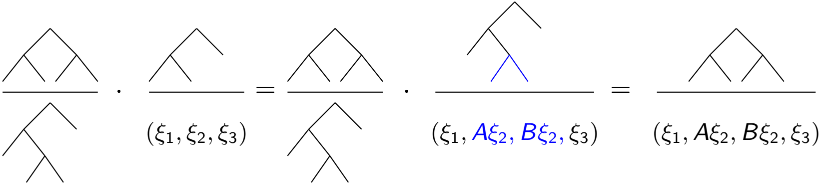

- Denote by $\sigma := \sigma_{A,B} : F \to \mathcal{U}(\mathscr{H}_{A,B})$, called the Pythagorean representation given by A, B.

Extending to the Cuntz Algebra

The Cuntz algebra $\mathcal{O} := \mathcal{O}_2$ is the universal C^*-algebra generated by two isometries s_1, s_2 such that $s_1 s_1^* + s_2 s_2^* = \mathrm{id}$ [Cuntz 77].

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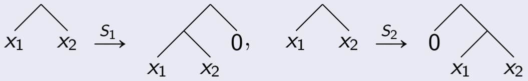
Proposition (Brothier-Jones 19, Brothier-W 23)

Every P-rep σ^F can be extended to a rep $\sigma^{\mathcal{O}}$ of $\mathcal{O}$.

Proof.

Define the isometries $S_1, S_2 \in B(\mathscr{H})$ with action given by:



It can be shown $S_1 S_1^* + S_2 S_2^* = \mathrm{id}$. Setting $\sigma^{\mathcal{O}}(s_i) = S_i$ gives a representation of $\mathcal{O}$. □

Examples of Pythagorean Representations [Brothier-Jones 19]

1. $\mathfrak{H} = \mathbf{C}, A = 1, B = 0$: $\sigma \cong 1_F \oplus \lambda_{F/F_{1/2}}$.

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4. $\mathfrak{H} = \mathbf{C}^2, A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} : \sigma \cong \lambda_{F/F_{1/3}}$.

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Definition (Brothier-W 23)

Let σ be a P-rep. The *Pythagorean dimension* $\dim_P(\sigma)$ is given by:

$$\dim_P(\sigma) = \min(\dim(A, B, \mathfrak{H}) : \sigma_{A,B} \cong \sigma).$$

How to Compute the Pythagorean Dimension?

Example:

$$A = \begin{pmatrix} 0 & 0 & 1/2 \\ 1 & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1/2 \\ 0 & 0 & 1/2 \end{pmatrix}, \ \mathfrak{H} = \mathbf{C}^3.$$

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Then $\mathfrak{H}_0 = \mathbf{C}e_1 \oplus \mathbf{C}e_2$, $\mathfrak{Z} = \mathbf{C}e_3$ with

$$A\restriction_{\mathfrak{H}_0} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ B\restriction_{\mathfrak{H}_0} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Hence $\dim(\mathfrak{H}) = 3$ but $\dim_P(\sigma) \leqslant 2$.

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Question 1: How can we compute the Pythagorean dimension?

How to Classify P-representations?

Proposition (Brothier-W 23)

Consider a finite-dim P-module $(A, B, \mathfrak{H})$ with P-rep $(\sigma, \mathscr{H})$. Then:

$$\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2 \oplus \cdots \oplus \mathfrak{H}_n \oplus \mathfrak{Z}$$

where $\mathfrak{H}_i$ are irreducible sub-modules and $\mathfrak{Z}$ the largest residual subspace. This induces a decomposition of $\mathscr{H}$ into subreps:

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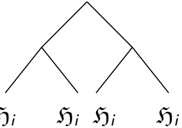

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Question 2: Are all subreps of σ in this form and when are they equivalent?

Diffuse Pythagorean Representations

Definition

1. A P-module $(A, B, \mathfrak{H})$ is *diffuse* if $\lim_{n\to\infty} p_n\xi = 0$ for all $\xi \in \mathfrak{H}$ and for all increasing sequences (p_n) of words in A, B.
2. A P-module is *atomic* if it does not contain any diffuse sub-modules.
3. A P-rep is *atomic* (*diffuse*) if from an atomic (diffuse) P-module.

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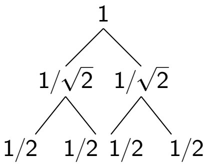
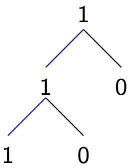


Example 2: $A = 1, B = 0$



Unique Smallest Complete Sub-module

Theorem (Brothier-W 23)

Let $(A, B, \mathfrak{H})$ *be a diffuse finite-dimensional P-module. Then there exists a unique smallest complete sub-module* $\mathfrak{K} \subset \mathfrak{H}$ *of* $\mathscr{H}$*.*
Moreover, $\dim_P(\sigma) = \dim(\mathfrak{K})$*.*

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Moreover, $\dim_P(\sigma) = \dim(\mathfrak{K})$.

Counter-example in infinite-dimensional case.
Let $\mathfrak{H} = \ell^2(\mathbf{Z})$ and $A = B = S/\sqrt{2}$ which is the unilateral shift operator divided by $\sqrt{2}$. Each subspace

$$\mathfrak{K}_j := \ell^2(\{j, j+1, j+2, \dots\})$$

with $j \in \mathbf{Z}$ defines a complete sub-module but clearly have trivial intersection.

Irreducibility of Diffuse P-representations

Theorem (Brothier-W 23)

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Example:

$$A = \begin{pmatrix} 1/\sqrt{2} & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/\sqrt{6} & 1/2 \\ 0 & 1/2 & 1/\sqrt{6} & -1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 1/\sqrt{2} & 0 & 0 & -1/2 \\ 0 & 1/2 & 0 & 0 \\ 0 & -1/2 & 2/\sqrt{6} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Equivalence of Diffuse P-representations

Theorem (Brothier-W 23)

Consider two full diffuse finite-dim P-modules $(A, B, \mathfrak{H})$ *and* $(A', B', \mathfrak{H}')$*. The associated P-reps are equivalent iff the P-modules are equivalent.*

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Counter-example in atomic case.
Take $\mathfrak{H} = \mathbf{C}^2$ and let

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$A' = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \; B' = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

Then $(A, B, \mathbf{C}^2)$, $(A', B', \mathbf{C}^2)$ are not equivalent P-modules but induce equivalent (irreducible) P-representations of F.

Manifold of Irreducible Classes

- $Irr_{diff}(d)$ the set of irreducible diffuse P-modules of P-dimension d;
- There is a group action of $PSU(d)$ on $Irr_{diff}(d)$ by

$$u \cdot (A, B, \mathbf{C}^d) = (uAu^*, uBu^*, \mathbf{C}^d).$$

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Theorem (Brothier-W 23)

1. *A given finite-dim P-module is almost surely diffuse and irreducible.*
2. $Irr_{diff}(d)$ *is a smooth submanifold of* $M_{2d,d}(\mathbf{C})$ *of real dim* $3d^2$*.*
3. $PSU(d)\backslash Irr_{diff}(d)$ *is a manifold of dimension* $2d^2 + 1$*.*
4. $PSU(d)\backslash Irr_{diff}(d)$ *is in bijection with the set of all irreducible classes of diffuse P-reps of P-dim* d*.*

Isomorphism of Categories

We have the below functors:

$$\mathrm{Mod}^{\mathrm{FD}}_{\mathrm{full,diff}}(\mathcal{P}) \to \mathrm{Rep}^{\mathrm{FD}}_{\mathrm{diff}}(F) \to \mathrm{Rep}^{\mathrm{FD}}_{\mathrm{diff}}(T) \to \mathrm{Rep}^{\mathrm{FD}}_{\mathrm{diff}}(V) \to \mathrm{Rep}^{\mathrm{FD}}_{\mathrm{diff}}(\mathcal{O}).$$

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Theorem (Brothier-W 23)

Let $X = F, T, V, \mathcal{O}$. Then:

1. $\mathrm{Mod}^{\mathrm{FD}}_{\mathrm{full,diff}}(\mathcal{P})$ *is equivalent to* $\mathrm{Rep}^{\mathrm{FD}}_{\mathrm{diff}}(X)$.
2. *The categories* $\mathrm{Rep}_{\mathrm{diff}}(X)$ *are isomorphic for* $X = F, T, V, \mathcal{O}$.
3. *The categories* $\mathrm{Rep}^{\mathrm{FD}}_{\mathrm{diff}}(X)$ *are semi-simple.*

Future Research Directions

1. Description of Pythagorean representations on classical Hilbert spaces.
2. Apply Jones' machinery to construct representations of Thompson-like groups.
3. Construction of a tensor product on the class of Pythagorean representations and studying the properties of the associated tensor category.