VOLTAGE AND DERIVED GRAPHS AND THEIR RELATION TO THE FREE PRODUCT OF GRAPHS

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Let G be a directed graph where each directed edge has been assigned an element from some group A, such that if an edge e is assigned $g \in A$, then the reverse edge $\overline{e}$ is assigned, $g^{-1} \in A$, the group element's inverse. The group A is called the *voltage group* and its elements are called *voltages*. The function, α, which maps edges to group elements is called the *voltage assignment*. The graph along with its voltage assignment are known as the *voltage graph*. The *derived graph* G^α of the voltage graph $\langle G, \alpha \rangle$ is the graph with vertex set $V(G) \times A$, edge set $E(G) \times A$, and with an edge from (u, a) to (v, ab) if and only if there is an edge from u to v in G with voltage b. The derived graph is a covering graph of the base graph G.

The free product, $\ast_{i=1}^n \Gamma_i$, is a covering graph of the Cartesian product $\prod_{i=1}^n \Gamma_i$ (a result from G. Willis, although we should also prove that u is locally bijective). The free product $\ast_{i=1}^n \Gamma_i$ can be is isomorphic to the derived graph of the voltage graph on $\prod_{i=1}^n \Gamma_i$ with the following voltage assignment α.

Choose a spanning tree, T, on $\prod_{i=1}^n \Gamma_i$. Assign the identity group element to each edge in T. For each of the remaining edges assign a unique free generator. Let the set of these generators be G. Now write relations, R, so that each copy of the Γ_i satisfy Kirchnoff's voltage law (we could also say 'each copy of the Γ_i *is balanced*' as is done in [1]), that is, in each copy of each Γ_i, the product of the group elements around every closed walk is the identity. Furthermore, these are the only relations on the group elements. Clearly, the voltage group used in this voltage graph is the group, A, with presentation $\langle G | R \rangle$. Note, this is not always the smallest presentation for the voltage group that can be used to construct the desired derived graph, in fact, in Proposition 0.3 we show that the voltage group is a free group.

Note that this voltage assignment is similar to the one used to get the universal covering of a graph (assign the identity to each edge in the spanning tree and a unique free generator to each edge not in the spanning tree), except that the relations glue parts together so cycles that stay in the same Γ_i in the cartesian product lift to cycles in the derived graph instead of just paths (as in the universal cover).

Definition 0.1. *Let* $c : E\left(\prod_{i=1}^n \Gamma_i\right) \to \{1, \dots, n\}$ *be the canonical colouring. We call a path* $e_1 \dots e_n$ *in* $\prod_{i=1}^n \Gamma_i$ monochrome *if there exists* i *with* $c(e_1) = \cdots = c(e_n) = i$.

Explain how we know there is only one connected component. The *net voltage* of a walk in G^α is the product of the group elements assigned to the edges in that walk. Let u be a vertex in G^α. The set of net voltages of walks that start and end at u form a subgroup of A, called the *local voltage group* at u, denoted $A(u)$. The local voltage group at every vertex in the voltage graph $\langle\prod_{i=1}^{n}\Gamma_i,\alpha\rangle$ is the entire voltage group A and therefore there is only one connected component in the derived graph (see cor 2, page 88 in [2]).

Give an isomorphism between the derived graph and the free product

Let p_v be a path from $\emptyset$ to v in $\ast_{i=1}^{n}\Gamma_i$ (note this goes through all the transition points). Consider the image of p_v under u in $\langle\prod_{i=1}^{n}\Gamma_i,\alpha\rangle$. Denote the net voltage (product of group elements) along $u(p_v)$ to be g_v. Note that if the Γ_i are not all trees, there can be multiple paths from $\emptyset$ to v in $\ast_{i=1}^{n}\Gamma_i$. However, all such paths must contain the same transition points (and in the same order) and hence they all enter and leave each Γ_i-sheet together and can only differ inside Γ_i-sheets.

Suppose in a Γ_i-sheet two distinct paths p and q start at o and end at t. Consider the image of p and q under u in $\langle\prod_{i=1}^{n}\Gamma_i,\alpha\rangle$. The walk $u(p)$ concatenated with walking in reverse along $u(q)$ is a closed walk in a single copy of a Γ_i and so the product of the group elements along it is the identity. Therefore the net voltage of $u(p)$ is equal to the net voltage of $u(q)$ and so the choice of path inside each Γ_i-sheet is irrelevant.

Proposition 0.2. *The map* $f : V(\ast_{i=1}^{n}\Gamma_i) \to V((\prod_{i=1}^{n}\Gamma_i)^{\alpha}) : v \mapsto (u(v), g_v)$ *is a graph isomorphism.*

Proof. f **is surjective**
Let $(\underline{w}, g) \in V((\prod_{i=1}^{n}\Gamma_i)^{\alpha})$. Then, since $g \in A = \langle G|R\rangle$ where G is the set of voltages on the edges and there is a spanning tree in $\langle\prod_{i=1}^{n}\Gamma_i,\alpha\rangle$ with identity voltage on each edge, there exists at least one path from $u(\emptyset)$ to $\underline{w}$ in $\langle\prod_{i=1}^{n}\Gamma_i,\alpha\rangle$ with net voltage g. To spell this out, suppose $g = g_1 g_2 \ldots . g_k$ where $g_i \in G$ then we can take the path from $u(\emptyset)$ along the spanning tree to the beginning of the edge with voltage g_1, then across this edge back to the spanning tree, then travel along the spanning tree to the beginning of the edge with voltage g_2 and continue this until you cross the edge with voltage g_n and then travel along the spanning tree to the vertex $\underline{w}$ (this path has net voltage g). This path in $\prod_{i=1}^{n}\Gamma_i$ lifts uniquely (by a standard algebraic topology theorem, once we show u is a covering map...need locally bijective) to a path in $\ast_{i=1}^{n}\Gamma_i$ via u (or do we say u^{-1}?) starting at $\emptyset$ and ending at a vertex x such that $f(x) = (\underline{w}, g)$.

f **is injective**
Suppose $f(x) = f(y)$. That is, $u(x) = u(y)$ and $g_x = g_y$. We wish to show that the path from $\emptyset$ to x has the same transition points as the path from $\emptyset$ to y. This along with $u(x) = u(y)$ is enough to conclude that $x = y$. Consider the closed walk $u(p_x)u(\overline{p_y})$ based at $u(x) = u(y)$. It has a net voltage of the identity so either $u(p_x) = u(p_y)$ which means $p_x = p_y$ and finally $x = y$, or $u(p_x)$ only differs with $u(p_y)$ in the path they take in each Γ_i copy (but enter and leave at the same vertex) and hence p_x and p_y have the same transition points, they end in the same Γ_i-sheet with $u(x) = u(y)$ and therefore $x = y$.

(a, b) **is an edge if and only if** $(f(a), f(b))$ **is an edge**
Suppose $(v, w) \in E(\ast_{i=1}^{n}\Gamma_i)$. Then, since $u : \ast_{i=1}^{n}\Gamma_i \to \prod_{i=1}^{n}\Gamma_i$ is a graph homomorphism (a result by G. Willis), $(u(v), u(w)) \in E(\prod_{i=1}^{n}\Gamma_i)$. Suppose the voltage assigned to $(u(v), u(w)) \in E(\prod_{i=1}^{n}\Gamma_i)$ by α is h. Then either $((u(v), g_v), (u(w), g_v h)$ or $((u(v), g_v), (u(w), g_v h^{-1})$ is an edge in the derived graph $(\prod_{i=1}^{n}\Gamma_i)^{\alpha}$. Now suppose $((u(v), a), (u(w), b)$

is an edge in the derived graph $(\prod_{i=1}^n \Gamma_i)^\alpha$. Therefore, there is an edge with voltage $a^{-1}b$ from $u(v)$ to $u(w)$ in $\langle \prod_{i=1}^n \Gamma_i, \alpha \rangle$. Since $u(v)$ is adjacent to $u(w)$ we know v is adjacent to w □

Proposition 0.3. *The voltage group on* $\prod_{i=1}^n \Gamma_i$ *which gives the derived graph isomorphic to* $*_{i=1}^n \Gamma_i$ *is isomorphic to the free group* $\mathbb{F}_m$ *where* $m = (n-1)\prod_{i=1}^n |V(\Gamma_i)| - \sum_{i=1}^n \prod_{j \neq i} |V(\Gamma_j)| + 1$

Proof. Consider the number of free generators we require. Choose any spanning tree T in $\prod_{i=1}^n \Gamma_i$. Only edges not in T could possibly receive a free generator as a voltage. But also, every independent monochromatic cycle must have one edge whose voltage cancels the rest of the voltage along the cycle so that it obeys Kirchnoff's voltage law and therefore this edge cannot be assigned a free generator. All other edges are assigned a distinct free generator. The number of independent cycles in a graph is known by many names and represents many things, but for a connected graph G it is $|E(G)| - |V(G)| + 1$. The number of edges in $\prod_{i=1}^n \Gamma_i$ that are not in T minus the number of independent monochromatic cycles in each copy of each Γ_i is

$$\sum_{i=1}^n |E(\Gamma_i)| \prod_{j \neq i} |V(\Gamma_j)| - \left(\prod_{i=1}^n |V(\Gamma_i)| - 1 \right) - \left(\sum_{i=1}^n (|E(\Gamma_i)| - |V(\Gamma_i)| + 1) \prod_{j \neq i} |V(\Gamma_j)| \right)$$

which simplifies to $(n-1)\prod_{i=1}^n |V(\Gamma_i)| - \sum_{i=1}^n \prod_{j \neq i} |V(\Gamma_j)| + 1$. Note we have not double counted any edges because there are no cycles in T. □

Note. The number of independent cycles for connected graphs is $r = |E(G)| - |V(G)| + 1$ (this is also known as the circuit rank, the minimum number of edges that must be removed from G to break all of its cycles,the cyclomatic number, the dimension of the cycle basis, the first Betti number of G, the rank of the first (integer) homology group rank$(H_1(G, \mathbb{Z}))$ and the number of ears in any 2-edge-connected graph).

References

[1] X. Chen, M.-A. Belabbas, and T. Başar, Cluster consensus with point group symmetries, *SIAM Journal on Control and Optimization*, **55(6)** 3869–3889.
[2] J. L. Gross and T. W. Tucker, *Topological Graph Theory*, John Wiley & Sons, 1987.