SEPARATING CYCLIC SUBGROUPS IN GRAPH PRODUCTS OF GROUPS

FEDERICO BERLAI AND MICHAL FEROV

Abstract. We prove that the property of being cyclic subgroup separable, that is having all cyclic subgroups closed in the profinite topology, is preserved under forming graph products.

Furthermore, we develop the tools to study the analogous question in the pro-p case. For a wide class of groups we show that the relevant cyclic subgroups - which are called p-isolated - are closed in the pro-p topology of the graph product. In particular, we show that every p-isolated cyclic subgroup of a right-angled Artin group is closed in the pro-p topology, and we fully characterise them.

Contents

1. Introduction

A very successful way to understand countable discrete groups certainly is by studying them through their finite quotients, and groups where this approach works to its full extent are called residually finite. A group G is *residually finite* if for any two distinct elements $g_1, g_2 \in G$ there is a finite group Q and a surjective homomorphism $\pi\colon G \to Q$ such that $\pi(g_1)$ is distinct from $\pi(g_2)$ in Q.

In general, properties of this type are called *separability properties*: a subset $S \subseteq G$ is said to be *separable* in G if for every $g \in G \setminus S$ there exists a finite quotient of G such that the image of g under the canonical projection does not belong to the image of S. This is equivalent to S being closed in the profinite topology of G. Therefore, a group is residually finite if the subset $\{e_G\}$ is separable.

One can then define a separability property by specifying which subsets are required to be separable: *conjugacy separable* groups have separable conjugacy classes of elements, LERF groups (also called locally extended residually finite) have separable finitely generated subgroups, and *cyclic subgroup separable* groups (also denoted CSS groups or π_c) are those where all cyclic subgroups are separable.

In a way, the notion of separability gives an algebraic analogue to decision problems in finitely presented groups: if the subset $S \subseteq G$ is given in suitably nice way (meaning that S is recursively enumerable and one can always effectively construct the image of S under the canonical projection onto a finite quotient of G) and it is separable in G, one can then decide whether a word in the generators of G represents an element belonging to S simply by checking finite quotients. Indeed, it was proved by Mal'cev [13] that finitely presented residually finite groups have solvable word problem, and Mostowski [17] showed that finitely presented conjugacy separable groups have solvable conjugacy problem. In a similar fashion, LERF groups have solvable generalised word problem, meaning that the membership problem is solvable for every finitely generated subgroup. In general, algorithms that involve enumerating finite quotients of an algebraic structure are called algorithms of Mal'cev-Mostowski type.

In this paper we study the already mentioned CSS groups, the groups in which all cyclic subgroups are separable. Naturally, one may describe CSS groups as the groups in which the *power problem*, i.e. given words u and v the problem of deciding whether u represents an element that is a power of the element represented by v, can be solved by an algorithm of Mal'cev-Mostowski type.

The main focus of this note is to study how does the property of being cyclic subgroup separable behave with respect to certain group-theoretic constructions. In particular, we study separability of cyclic subgroups in *graph products of groups*, a natural generalisation of free and direct products in the category of groups, first introduced by Green [8]. Let $\Gamma = (V\Gamma, E\Gamma)$

be a simplicial graph, i.e. $V\Gamma$ is a set and $E\Gamma \subseteq \binom{V\Gamma}{2}$, and let $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be a collection of groups. The graph product $\Gamma\mathcal{G}$ is the quotient of the free product $*_{v\in V\Gamma} G_v$ modulo all the relations of the form

$$g_u g_v = g_v g_u \text{ for all } g_u \in G_u, g_v \in G_v, \{u, v\} \in E\Gamma.$$

The groups $G_v \in \mathcal{G}$ are referred to as *vertex groups*. Well-known examples of graph products are *right-angled Artin groups* (raags) and *right-angled Coxeter groups*, where all vertex groups are respectively infinite cyclic, or cyclic groups of order two.

When Γ is totally disconnected, that is the edge set is empty, the resulting graph product is the free product of the vertex groups, and if Γ is complete, that is any two vertices are joined by an edge, the resulting graph product is the direct sum of the vertex groups.

Green proved that the class of residually finite groups is closed with respect to forming graph products [8, Corollary 5.4]. On the other hand, Minasyan proved that raags are conjugacy separable [15, Theorem 1.1], and this result was later generalised by the second-named author to show that the class of conjugacy separable groups is closed under forming graph products [7, Theorem 1.1].

Closure properties of the class of CSS groups have been studied previously. Burillo and Martino [5] studied the class of quasi-potent CSS groups. In particular, they show that the class of quasi-potent groups is closed with respect to special HNN-extensions along a separable subgroup, amalgamations along a virtually-cyclic subgroup, and direct products. Bobrovskii and Sokolov [3] showed that the class of CSS groups is closed under amalgamation along a common retract.

The first result of this work is to prove that the class of cyclic subgroup separable groups is closed under forming graph products. This generalises a result of Green, who showed that right-angled Artin groups are CSS [8, Theorem 2.16].

Theorem A. The class of CSS groups is closed under forming graph products.

Let us stress that Theorem A cannot be strengthened to LERF, that is to groups where all finitely generately subgroups are closed in the profinite topology. Indeed, the class of LERF groups is not closed under forming direct products: free groups are LERF [9], but the group $F_2 \times F_2$ contains finitely generated subgroups with unsolvable membership problem [14], hence it cannot be LERF.

The notion of separability can be generalised in a natural way by considering only certain kind of quotients: let $\mathcal{C}$ be a class of groups, we then say that a subset $S \subseteq G$ is $\mathcal{C}$-separable in G if for every $g \in G \setminus S$ there is a group $Q \in \mathcal{C}$ and a epimorphism $\pi\colon G \to Q$ such that $\pi(g)$ does not belong to $\pi(S)$.

Toinet [22] extended Minasyan's result about conjugacy separability in raags, showing that conjugacy classes in raags are separable with respect to the class of all finite p-groups. Toinet's result was later generalised by the second-named author [7], who proved that the class of $\mathcal{C}$-conjugacy separable groups is closed under forming graph products whenever $\mathcal{C}$ is an extension closed pseudovariety of finite groups.

The original result of Green on residual finiteness being preserved by graph products was extended by the authors [2, Theorem A], where it was shown that the class of residually $\mathcal{C}$ groups is closed with respect to forming graph products for many classes $\mathcal{C}$ of groups, including solvable or amenable groups.

After proving Theorem A, we shift our attention to cyclic subgroup separability in the context of pro-p topologies. We consider the class of all finite p-groups, where p is some prime number, and we say that a subset $S \subseteq G$ is p-separable in G if it is closed in the pro-p topology of G. This case is subtler than the one of profinite topology: all subgroups of $\mathbb{Z} = \langle a \rangle$ are closed in the profinite topology, but $\langle a^2 \rangle$ is not p-separable in $\mathbb{Z}$ for any odd prime p.

Following [1, 3] (compare also Definition 7.2) we say that the subgroup $H \leq G$ is *p-isolated* in G if for any $g \in G$ and any prime q distinct from p the following implication holds: whenever $g^q \in H$, then already $g \in H$. We say that G is *cyclic subgroup p-separable* (p-CSS for short) if all its p-isolated cyclic subgroups are p-separable.

To be able to easily identify p-isolated subgroups, we develop the notion of primitive stability in the context of groups with unique roots (see respectively Definition 7.5 and Definition 6.1). Roughly speaking, G is a primitively stable group with unique roots if expressing elements of G as powers of "smaller" elements behaves in a predictable manner, similar to infinite cyclic groups. We refer to Section 7 for the precise definition of primitive stability and its connection to p-isolation.

To ease the notation, let $\mathfrak{U}_{\mathrm{ps}}$ denote the class of primitively stable groups with unique roots. After showing that $\mathfrak{U}_{\mathrm{ps}}$ is closed under taking graph products (compare Theorem 8.6), we prove the analogous of Theorem A for pro-p topologies.

Theorem B. For every prime number p, the class of p-CSS groups in $\mathfrak{U}_{\mathrm{ps}}$ is closed under forming graph products.

As the infinite cyclic group belongs to $\mathfrak{U}_{\mathrm{ps}}$ and it is p-CSS, we obtain:

Corollary B. For every prime number p, right-angled Artin groups are p-CSS.

Groups in the class $\mathfrak{U}_{\mathrm{ps}}$ admit a more algebraic characterisation. Let $g \in G$, and define the *radical* of g in G as the subset

$$\mathrm{Rad}_G(g) = \big\{ r \in G \mid r^a \in \langle g \rangle \text{ for some } a \in \mathbb{Z} \setminus \{0\} \big\}.$$

The radical of an element $g \in G$ is not in general a subgroup, but this is the case for groups in $\mathfrak{U}_{\mathrm{ps}}$. Moreover, we characterise the groups in $\mathfrak{U}_{\mathrm{ps}}$ as the torsion-free groups where $\mathrm{Rad}_G(g)$ is a cyclic subgroup for every non-trivial element.

Theorem C. Let G be a torsion-free group. The following are equivalent:

(1) $G \in \mathfrak{U}_{\mathrm{ps}}$;
(2) for all non-trivial $g \in G$ the subset $\mathrm{Rad}_G(g)$ is an infinite cyclic subgroup of G.

Exploiting Theorem C, we prove that the class $\mathfrak{U}_{\mathrm{ps}}$ is quite wide: torsion-free hyperbolic groups, (residually) finitely generated torsion-free nilpotent groups, and torsion-free groups hyperbolic relative to groups in $\mathfrak{U}_{\mathrm{ps}}$, all belong to $\mathfrak{U}_{\mathrm{ps}}$. In particular, limit groups belong to $\mathfrak{U}_{\mathrm{ps}}$, being toral relatively hyperbolic. As already stated, the class $\mathfrak{U}_{\mathrm{ps}}$ is closed under graph products.

The paper is organised as follows. In Section 2 we recall the notion of profinite and pro-$\mathcal{C}$ topologies on groups, and review the classical results that allow us to use topological methods when working with separability properties; readers familiar with pro-$\mathcal{C}$ topologies might feel free to skip this section. In Section 3 we recall the notation for graph products of groups and review known properties, such as Normal Form Theorem and the definition of cyclically reduced elements and full subgroups.

In Section 4 we introduce the notion of amalgams over a common retract, develop the standard combinatorial framework for this type of free-construction and show how it relates to graph products in a natural way: we show that, as soon as it is not a direct product of its vertex groups, a graph product splits as an amalgam of its proper full subgroups over a common retract. Using the previously developed framework, in Section 5 we prove Theorem A.

In Section 6 we recall the notion of Unique Roots property and, using the framework for amalgams over retracts, we show that the the class of groups with Unique Roots property - which we denote by $\mathfrak{U}$ - is closed under forming graph products. In Section 7 we introduce the notion of primitive logarithms, primitive roots and primitive stability. In the class $\mathfrak{U}_{\mathrm{ps}}$ of primitively stable groups with unique roots we can give a characterisation of p-isolated cyclic subgroups in terms of the primitive logarithm of the generator of the cyclic subgroup.

In Section 8 we study the closure properties of the class $\mathfrak{U}_{\mathrm{ps}}$. In particular, we show that it is closed with respect to taking subgroups, direct products, amalgams over retracts and, consequently, graph products. In Section 9 we prove Theorem C, and we show that the class $\mathfrak{U}_{\mathrm{ps}}$ contains finitely generated residually torsion-free nilpotent groups, torsion-free hyperbolic groups and certain relatively hyperbolic groups. A proof of Theorem B is given in Section 10. Finally, in Section 11 we give a characterisation of p-isolated cyclic subgroups in graph products of groups.

2. Pro-$\mathcal{C}$ topologies on groups

This section, after a brief paragraph on notation, contains basic facts about pro-$\mathcal{C}$ topologies on groups; we are including it to make the paper self-contained and experts can feel free to skip it. Proofs of all the statements can be found in the classic book by Ribes and Zalesskii [21] or in the second named author's thesis [6].

If G is a group, then e_G, or e when the group G is clear from the context, denotes the identity element in G. For elements $g, h \in G$ we will use g^h to denote hgh^{-1}, the h-conjugate of g. Similarly, for a subgroup $H \leq G$ we will use g^H to denote $\{hgh^{-1} \mid h \in H\}$. In this note the natural numbers $\mathbb{N}$ include zero.

Let $\mathcal{C}$ be a class of groups and let G be a group. We say that a normal subgroup $N \trianglelefteq G$ is a *co-$\mathcal{C}$ subgroup* of G if $G/N \in \mathcal{C}$, and we denote by $\mathcal{N}_{\mathcal{C}}(G)$ the set of co-$\mathcal{C}$ subgroups of G.

Consider the following closure properties for a class of groups $\mathcal{C}$:

(c0) $\mathcal{C}$ is closed under taking finite subdirect products,
(c1) $\mathcal{C}$ is closed under taking subgroups,
(c2) $\mathcal{C}$ is closed under taking finite direct products.

Note that

$$(c0) \Rightarrow (c2) \qquad \text{and} \qquad (c1) + (c2) \Rightarrow (c0).$$

If the class $\mathcal{C}$ satisfies (c0) then, for every group G, the set $\mathcal{N}_{\mathcal{C}}(G)$ is closed under finite intersections, that is to say, if $N_1, N_2 \in \mathcal{N}_{\mathcal{C}}(G)$ then also $N_1 \cap N_2 \in \mathcal{N}_{\mathcal{C}}(G)$. This implies that $\mathcal{N}_{\mathcal{C}}(G)$ is a base at e_G for a topology on G.

Hence the group G can be equipped with a group topology, where the base of open sets is given by

$$\{gN \mid g \in G, N \in \mathcal{N}_{\mathcal{C}}(G)\}.$$

This topology, denoted by pro-$\mathcal{C}(G)$, is called the *pro-$\mathcal{C}$ topology* on G.

If the class $\mathcal{C}$ satisfies (c1) and (c2), or equivalently, (c0) and (c1), then one can easily see that equipping a group G with its pro-$\mathcal{C}$ topology is a faithful functor from the category of groups to the category of topological groups.

Lemma 2.1. *Let $\mathcal{C}$ be a class of groups satisfying (c1) and (c2). Given groups G and H, every homomorphism $\varphi\colon G \to H$ is a continuous map with respect to the corresponding pro-$\mathcal{C}$ topologies. Furthermore, if φ is an isomorphisms, then it is a homeomorphism.*

A set $X \subseteq G$ is *$\mathcal{C}$-closed* in G if X is closed in pro-$\mathcal{C}(G)$: for every $g \notin X$ there exists $N \in \mathcal{N}_{\mathcal{C}}(G)$ such that the open set gN does not intersect X, that is, $gN \cap X = \emptyset$. This is equivalent to $gN \cap XN = \emptyset$, and hence $\varphi(g) \notin \varphi(X)$ in G/N, where $\varphi\colon G \twoheadrightarrow G/N$ is the canonical projection onto the quotient G/N. Accordingly, a set is *$\mathcal{C}$-open* in G if it is open in pro-$\mathcal{C}(G)$.

In this paper we will only consider the class $\mathcal{C}$ of all finite groups or of all finite p-groups, and therefore we will assume that $\mathcal{C}$ is closed under subgroups, finite direct products, quotients and extensions. A class of groups satisfying these properties is also called an extension-closed pseudovariety of finite groups.

In the following lemma we collect known facts about open and closed subgroups, in particular [10, Theorem 3.1, Theorem 3.3].

Lemma 2.2. *Let G be a group and let $H \leq G$. Then*

(i) *H is $\mathcal{C}$-open in G if and only if there is $N \in \mathcal{N}_{\mathcal{C}}(G)$ such that $N \leq H$; moreover, every $\mathcal{C}$-open subgroup is of finite index in G and it is $\mathcal{C}$-closed in G;*

(ii) *H is $\mathcal{C}$-closed in G if and only if H is an intersection of open subgroups.*

Remark 2.3. Let G_1, G_2 be groups and suppose that $H_1 \leq G_1$, $H_2 \leq G_2$ are $\mathcal{C}$-closed in G_1 and G_2 respectively. Then $H_1 \times H_2$ is $\mathcal{C}$-closed in $G_1 \times G_2$.

2.1. **Restrictions of pro-$\mathcal{C}$ topologies.** Let G be a group and let $H \leq G$. We say that that pro-$\mathcal{C}(H)$ is a *restriction* of pro-$\mathcal{C}(G)$ if a subset $X \subseteq H$ is $\mathcal{C}$-closed in H if and only if it is $\mathcal{C}$-closed in G. Note that if pro-$\mathcal{C}(H)$ is a restriction of pro-$\mathcal{C}(G)$ then H is $\mathcal{C}$-closed in G as H is $\mathcal{C}$-closed in H by definition.

Lemma 2.4. *Let G be a group and let $H \leq G$ be a subgroup. Then* pro-$\mathcal{C}(H)$ *is a restriction of* pro-$\mathcal{C}(G)$ *if and only if every $N \in \mathcal{N}_{\mathcal{C}}(H)$ is $\mathcal{C}$-closed in G.*

Proof. Suppose that pro-$\mathcal{C}(H)$ is a restriction of pro-$\mathcal{C}(G)$ and let $N \in \mathcal{N}_{\mathcal{C}}(H)$ be arbitrary. Clearly, N is $\mathcal{C}$-closed in H and thus it is $\mathcal{C}$-closed in G.

Now suppose that every $N \in \mathcal{N}_{\mathcal{C}}(H)$ is $\mathcal{C}$-closed in G and let $X \subseteq H$ be $\mathcal{C}$-closed in H. Obviously, for every $g \in H$ there is some $N_g \in \mathcal{N}_{\mathcal{C}}(H)$ such that $g \notin XN_g$. As $|H : N_g| < \infty$ we see that there are $g_1, \ldots, g_n \in X$ such that $XN_g = \bigcup_{i=1}^{n} g_i N_g$. This means that XN_g is a finite union of sets $\mathcal{C}$-closed in G and therefore it is $\mathcal{C}$-closed in G. In particular, we see that $X = \cap_{g \in H \setminus X} XN_g$ is an intersection of $\mathcal{C}$-closed sets in G, hence it is $\mathcal{C}$-closed in G. $\square$

One can easily show the following by using the proof of [21, Lemma 3.1.5]

Lemma 2.5. *Let G be a residually $\mathcal{C}$ group and suppose that $R \leq G$ is a retract. Then* pro-$\mathcal{C}(R)$ *is a restriction of* pro-$\mathcal{C}(G)$.

Lemma 2.6. *Let G_1, G_2 be groups and let $H_1 \leq G_1$, $H_2 \leq G_2$ be their subgroups. Suppose that* pro-$\mathcal{C}(H_1)$ *is a restriction of* pro-$\mathcal{C}(G_1)$ *and that* pro-$\mathcal{C}(H_2)$ *is a restriction of* pro-$\mathcal{C}(G_2)$*. Then* pro-$\mathcal{C}(H_1 \times H_2)$ *is a restriction of* pro-$\mathcal{C}(G_1 \times G_2)$.

Proof. Following Lemma 2.4 we see that H_1 is $\mathcal{C}$-closed in G_1, similarly H_2 is $\mathcal{C}$-closed in G_2. Let $N \in \mathcal{N}_{\mathcal{C}}(H_1 \times H_2)$ be arbitrary. Set $N_1 = N \cap H_1$

and $N_2 = N \cap N_2$. As the class $\mathcal{C}$ is closed under taking subgroups, we see that $N_1 \in \mathcal{N}_{\mathcal{C}}(H_1)$ and $N_2 \in \mathcal{N}_{\mathcal{C}}(H_2)$. As $\mathcal{C}$ is closed under taking direct products, $N_1 \times N_2 \in \mathcal{N}_{\mathcal{C}}(H_1 \times H_2)$ and therefore $N_1 \times N_2$ is of finite index in $H_1 \times H_2$ by Lemma 2.2. Consequently, $N_1 \times N_2$ is of finite index in N.

As N_1 is $\mathcal{C}$-closed in H_1 and pro-$\mathcal{C}(H_1)$ is a restriction of G_1, we see that N_1 is $\mathcal{C}$-closed in G_1. Similarly, N_2 is $\mathcal{C}$-closed in G_2. Using Remark 2.3 we see that $N_1 \times N_2$ is $\mathcal{C}$-closed in $G_1 \times G_2$. As $N_1 \times N_2$ is a subgroup of finite index in N we see that N is $\mathcal{C}$-closed in $G_1 \times G_2$. It follows by Lemma 2.4 that pro-$\mathcal{C}(H_1 \times H_2)$ is a restriction of pro-$\mathcal{C}(G_1 \times G_2)$. □

3. Graph products

We recall here some terminology and facts about graph products that will be used in this paper. Let $G = \Gamma\mathcal{G}$ be a graph product. Every element $g \in G$ can be obtained as a product of a sequence $W \equiv (g_1, g_2, \dots, g_n)$, where each g_i belongs to some $G_{v_i} \in \mathcal{G}$. We say that W is a *word* in G and that the elements g_i are its *syllables*. The *length* of a word is the number of its syllables, and it is denoted by $|W|$.

Transformations of the three following types can be defined on words in graph products:

- (T1) remove the syllable g_i if $g_i = e_{G_v}$, where $v \in V$ and $g_i \in G_v$,
- (T2) remove two consecutive syllables g_i, g_{i+1} belonging to the same vertex group G_v and replace them by the single syllable $g_i g_{i+1} \in G_v$,
- (T3) interchange two consecutive syllables $g_i \in G_u$ and $g_{i+1} \in G_v$ if $\{u, v\} \in E$.

The last transformation is also called *syllable shuffling*. Note that transformations (T1) and (T2) decrease the length of a word, whereas transformation (T3) preserves it. Thus, applying finitely many of these transformations to a word W, we can obtain a word W' which is of minimal length and that represents the same element in G.

For $1 \leq i < j \leq n$, we say that syllables g_i, g_j can be *joined together* if they belong to the same vertex group and 'everything in between commutes with them'. More formally: $g_i, g_j \in G_v$ for some $v \in V$ and for all $i < k < j$ we have that $g_k \in G_{v_k}$ for some $v_k \in \mathrm{link}(v) := \{u \in V \mid \{u, v\} \in E\}$. In this case the words

$$W \equiv (g_1, \dots, g_{i-1}, g_i, g_{i+1}, \dots, g_{j-1}, g_j, g_{j+1}, \dots, g_n)$$

and

$$W' \equiv (g_1, \dots, g_{i-1}, g_i g_j, g_{i+1}, \dots, g_{j-1}, g_{j+1}, \dots, g_n)$$

represent the same group element in G, and the length of the word W' is strictly shorter than W.

We say that a word $W \equiv (g_1, g_2, \dots, g_n)$ is *reduced* if it is either the empty word, or if $g_i \neq e$ for all i and no two distinct syllables can be joined together. As it turns out, the notion of being reduced and the notion of being of minimal length coincide, as it was proved by Green [8, Theorem 3.9]:

Theorem 3.1 (Normal Form Theorem)**.** *Every element g of a graph product G can be represented by a reduced word. Moreover, if two reduced words W, W' represent the same element in the group G, then W can be obtained from W' by a finite sequence of syllable shufflings. In particular, the length of a reduced word is minimal among all words representing g, and a reduced word represents the trivial element if and only if it is the empty word.*

Thanks to Theorem 3.1 the following are well defined. Let $g \in G$ and let $W \equiv (g_1, \dots, g_n)$ be a reduced word representing g. We define the *length* of g in G to be $|g| = n$ and the *support* of g in G to be

$$\mathrm{supp}(g) = \{v \in V | \exists i \in \{1, \dots, n\} \text{ such that } g_i \in G_v \setminus \{e\}\}.$$

We define $\mathrm{FL}(g) \subseteq V\Gamma$ as the set of all $v \in V\Gamma$ such that there is a reduced word W that represents the element g and starts with a syllable from G_v. Similarly, we define $\mathrm{LL}(g) \subseteq V\Gamma$ as the set of all $v \in V\Gamma$ such that there is a reduced word W that represents the element g and ends with a syllable from G_v. Note that $\mathrm{FL}(g) = \mathrm{LL}(g^{-1})$.

Let $x, y \in G$ and let $W_x \equiv (x_1, \dots, x_n), W_y \equiv (y_1 \dots, y_m)$ be reduced expressions for x and y, respectively. We say that the product xy is a *reduced product* if the word $(x_1, \dots, x_n, y_1, \dots, y_m)$ is reduced. Obviously, xy is a reduced product if and only if $|xy| = |x| + |y|$ or, equivalently, if $\mathrm{LL}(x) \cap \mathrm{FL}(y) = \emptyset$. We can naturally extend this definition: for $g_1, \dots, g_n \in G$ we say that the product $g_1 \dots g_n$ is reduced if $|g_1 \dots g_n| = |g_1| + \dots + |g_n|$.

Every subset of vertices $X \subseteq V\Gamma$ induces a full subgraph Γ_X of the graph Γ. Let G_X be the subgroup of G generated by the vertex groups corresponding to the vertices contained in X. Subgroups of G that can be obtained in such way are called *full subgroups* of G; according to standard convention, $G_\emptyset = \{e\}$. We say that G_X is a *proper full subgroup* if $G_X \neq G$.

Using the normal form theorem it can be seen that G_X is naturally isomorphic to the graph product of the family $\mathcal{G}_X = \{G_v \mid v \in X\}$ with respect to the full subgraph Γ_X. For these subgroups, there exists a canonical retraction $\rho_X \colon G \to G_X$ defined on the standard generators of G as follows:

$$\rho_X(g) = \begin{cases} g & \text{if } g \in G_v \text{ for some } v \in X, \\ 1 & \text{otherwise.} \end{cases}$$

We will often abuse the notation and sometimes consider the retraction ρ_X as a surjective homomorphism $\rho_X \colon G \to G_X$, and sometimes as an endomorphism $\rho_X \colon G \to G$. In that case writing $\rho_X \circ \rho_Y$, where $Y \subseteq V\Gamma$, makes sense.

Let $A, B \subseteq V\Gamma$ be arbitrary, $G_A, G_B \leq G$ be the corresponding full subgroups of G, and let ρ_A, ρ_B be the corresponding retractions. Then ρ_A and ρ_B commute: $\rho_A \circ \rho_B = \rho_B \circ \rho_A$. It follows that $G_A \cap G_B = G_{A \cap B}$ and $\rho_A \circ \rho_B = \rho_{A \cap B}$.

3.1. **Cyclically reduced elements.** Let $g \in \Gamma\mathcal{G}$ be an element of a graph product, and let $W \equiv (g_1, \dots, g_n)$ be a reduced word representing it. We say that g is *cyclically reduced* if all cyclic permutations $(g_{i+1}, \dots, g_n, g_1, \dots, g_i)$ of W, for $i = 1, \dots, n-1$, are reduced words. In view of [7, Lemma 3.8] this definition is well posed, because it is independent of the choice of the reduced word W representing the element g.

Lemma 3.2. *Let $G = \Gamma\mathcal{G}$ and let $g \in G$ be an arbitrary element. The following are equivalent:*

(i) *g is cyclically reduced,*
(ii) *$|g| \leq |f|$ for every $f \in g^G$.*

Proof. If $|g| = 1$ then the statement is true. Suppose therefore that $|g| > 1$ and that g is not cyclically reduced. Then, there exists a reduced word $W \equiv (g_1, \dots, g_n)$ representing g, and a cyclic permutation of W is not reduced. Therefore, it must be that g_1 and g_n are elements that belong to the same vertex group of $\Gamma\mathcal{G}$. This implies that

$$|g_1^{-1} g g_1| \leq n - 1 < n = |g|,$$

so that condition (ii) is not satisfied.

Suppose now that there exists an element $h \in G$ such that $|h^{-1}gh| < |g|$, let $W_h \equiv (h_1, \dots, h_r)$ be a reduced word representing it, and let $W_g \equiv (g_1, \dots, g_n)$ be a reduced word representing g. As $|h^{-1}gh| < |g|$, it must in particular be (up to syllable shuffling in W_h and in W_g) that the elements g_1, g_n and h_1 belong to the same vertex group.

Therefore, there exists a shuffling of W_g whose first and the last element belong to the same vertex group. From [7, Lemma 3.8] we conclude that the element g is not cyclically reduced. $\square$

Lemma 3.2 implies that, given any element $g \in G$, there always exists a cyclically reduced $g' \in g^G$.

We define the *essential support* of an element $g \in G$, denoted by $\operatorname{esupp}(g)$, to be the support $\operatorname{supp}(g')$ of a cyclically reduced element $g' \in g^G$.

4. Amalgams over retracts

Suppose that $G_A = K_A \rtimes R$, $G_A = K_A \rtimes R$ are semidirect products and let $\rho_A \colon G_A \to R$, $\rho_B \colon G_B \to R$ be the corresponding retractions. We say that R is a *common retract* for G_A and G_B. Consider the amalgamated free product $G = G_A *_R G_B$, where the amalgamation is taken along the identity map $\mathrm{id}_R \colon R \to R$. We say that G is an *amalgam over a retract*, and we have that

$$G = (K_A \rtimes R) *_R (K_B \rtimes R) \cong (K_A * K_B) \rtimes R. \tag{1}$$

From Equation (1), we see that every $g \in G$ can be expressed as a product $g = k_1 \dots k_m r$ for some elements $k_1, \dots, k_m \in K_A \cup K_B$ and $r \in R$. We say that this expression is *reduced* if $k_i \neq e$ for all $i = 1, \dots, m$ and k_i, k_{i+1} do not belong the same factor for $i = 1, \dots, m-1$. If $m = 0$ and $r = e$, we say that the expression is *trivial*, otherwise it is *non-trivial*.

We will use the following fact:

Lemma 4.1 (Normal form theorem for amalgams over retracts)**.** *Let $G_A = K_A \rtimes R$, $G_B = K_B \rtimes R$ be groups with a common retract and let $G = G_A *_R G_B$ be the corresponding amalgam over the retract R. For every $g \in G$ the corresponding reduced expression is given uniquely, and a reduced expression represents the trivial element in G if and only if it is trivial.*

Let $g \in G$ be arbitrary. With $|g|_*$ we denote the free-product length of the factor of g belonging to $K_1 * K_2$, that is, if $g = k_1 \dots k_n r$ is a reduced expression for the element g, then $|g|_* = n$. This is the word length in $K_1 * K_2$ with respect to the (potentially infinite) generating set $K_1 \cup K_2$.

Note that if $|g|_* > 0$ then $g \neq e$, but the opposite implication does not hold: $|r|_* = 0$ for every $r \in R$.

4.1. **Graph products as amalgams over retracts.** Let Γ be a graph and $C \subseteq V\Gamma$. We say that the subset C is *separating* if the induced subgraph $\Gamma_{V\Gamma \setminus C}$ has at least two connected components. Therefore, the graph Γ contains a separating set if and only if Γ contains a pair of distinct vertices $u, v \in V\Gamma$ such that $\{u, v\} \notin E\Gamma$, i.e. if Γ is not complete. Note that, by definition, if the graph Γ is disconnected then the empty set $\emptyset \subseteq V\Gamma$ is a separating set in Γ. In fact, $\emptyset$ is separating if and only if Γ is disconnected.

Lemma 4.2. *Let Γ be a graph and let $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be a family of non-trivial groups. If Γ contains a separating set, then $G = \Gamma\mathcal{G}$ properly splits as an amalgam over a retract, where the factors (and the amalgamated subgroup) are full subgroups.*

Proof. Suppose that $C \subset V\Gamma$ is a (potentially empty) separating set of vertices, so that the graph $\Gamma \setminus C$ has at least two connected components. Write $V\Gamma \setminus S = A' \sqcup B'$, where A' is one of these connected components, and B' consists of all remaining vertices. It follows that $G \cong G = G_A *_{G_C} G_B$, where $A = A' \cup C$ and $B = B' \cup C$.

The subgroups G_A, G_B and G_C are retracts of G because they all are full subgroups. For the same reason, G_C is a retract of both G_A and G_B. □

Note that the choice of the splitting in Lemma 4.2 is not unique, and depends both on the choice of the separating subset C, and on how to express the set $V\Gamma \setminus C$ as the union of two sets given by its connected components.

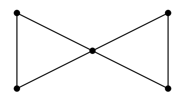
For a vertex $v \in V\Gamma$ we define the *link* $\operatorname{link} v$ to be the set of vertices adjecient to v in$V\Gamma$, and the *star* to be $\operatorname{star}(v) = \operatorname{link}(v) \cup \{v\}$. For a subset $A \subseteq V\Gamma$, we define $\operatorname{star}(A)$ to be $\bigcap_{v \in A} \operatorname{star}(v)$.

Separating subsets can be obtained using links of vertices: if $v \in V\Gamma$ is a vertex that is not connected by an edge to every other vertex of Γ, i.e.

if $\operatorname{link}(v) \neq V\Gamma \setminus \{v\}$, then $\operatorname{link}(v)$ is a separating subset, and the induced splitting is

$$G \cong G_{V\Gamma\setminus\{v\}} *_{G_{\operatorname{link}(v)}} \left(G_{\operatorname{link}(v)} \times G_v\right).$$

We did not use this fact in the proof of Lemma 4.2, because it might happen that the minimal separating subset in a graph cannot be expressed as the link of a vertex, as for instance in the following graph:



4.2. **Cyclically reduced elements.** Let $g \in (K_1 * K_2) \rtimes R$ and let $g = k_1 \dots k_l r$ be a reduced expression for g. We say that g is *cyclically reduced* if either one of the following is true:

(i) $l \in \{0, 1\}$,
(ii) $l \geq 2$ and k_1, k_l do not belong to the same factor.

Note that if $|g|_*$ is even then g is necessarily cyclically reduced. If $|g|_* = 2m + 1$ for some $m \in \mathbb{N}$, then g is cyclically reduced if and only if $|g|_* = 1$.

If $g = k_1 \dots k_l r$ is a reduced expression for an element $g \in (K_1 * K_2) \rtimes R$, we say that $c \in K_1 * K_2$ is a *prefix* of g if $c = k_1 \dots k_{l'}$ for $l' \leq l$.

Lemma 4.3. *Let $g \in (K_1 * K_2) \rtimes R$ be not cyclically reduced with $|g|_* = 2m+1$, where $m \geq 1$. There exists a prefix c of g such that $c^{-1}gc$ is cyclically reduced and $|c|_* \leq m$.*

Proof. Let $g = k_1 \dots k_{2m+1} r$ be the reduced expression for g. By assumption, k_1 and k_{2m+1} belong to the same factor, and moreover k_i and k_{2m+2-i} belong to the same factor, for all $i = 1, \dots, m$.

Suppose that that there exists $0 < m' < m$ such that $(k_{2m+2-m'} k_{m'}^r) \neq 1$, let m' be the smallest natural possible with this property, and set $c = k_1 \dots k_{m'}$. Then

$$\begin{aligned} c^{-1}gc &= (k_1 \dots k_{m'})^{-1}(k_1 \dots k_{2m+1} r)(k_1 \dots k_{m'}) \\ &= k_{m'+1} \dots k_{2m+1-m'}(k_{2m+2-m'} k_{m'}^r) r. \end{aligned} \tag{2}$$

As the element $(k_{2m+2-m'} k_{m'}^r)$ is not trivial, the expression of Equation (2) is reduced. Moreover, the elements $k_{m'+1}$ and $k_{m'}$ (and therefore $k_{m'+1}$ and $(k_{2m-m'} k_{m'}^r)$) belong to different factors. We therefore see that the element $c^{-1}gc$ is cyclically reduced.

If no such m' exists, set $c = k_1 \dots k_m$. We have that

$$c^{-1}gc = (k_1 \dots k_m)^{-1}(k_1 \dots k_{2m+1} r)(k_1 \dots k_m) = k_{m+1} r$$

is cyclically reduced. □

Lemma 4.4. *For $i = 1, 2$ let $G_i = K_i \rtimes R$, consider the amalgam $G = G_1 *_R G_2 \cong (K_1 * K_2) \rtimes R$, and suppose that for every $g \in G_i \setminus R$ we have $\langle g \rangle \cap R = \{e\}$. Let $g \in G$ and $n \in \mathbb{Z} \setminus \{0\}$ be arbitrary. Then g^n is cyclically reduced if and only if g is cyclically reduced.*

Furthermore, in this case, we have that $|g^n|_* = 1$ if and only if $|g|_* = 1$, and $|g^n|_* = |n| \cdot |g|_*$ otherwise.

Proof. Let $g = k_1 \dots k_m r$ be the reduced expression for g and suppose that g is cyclically reduced. There are two cases to consider: either $m \in \{0, 1\}$, or $m = 2l$ for some $l \geq 1$.

If $m \in \{0, 1\}$, then g belongs to one of the factors K_i, and therefore g^n is cyclically reduced as well.

Suppose that $m = 2l$ for some $l \geq 1$. We have that

$$g^n = k_1 \dots k_{2l} k_1^r \dots k_{2l}^r \dots k_1^{r^{n-1}} \dots k_{2l}^{r^{n-1}} r^n \tag{3}$$

is the reduced expression for g^n, and therefore g^n is cyclically reduced.

Now assume that g is not cyclically reduced, so that $m = 2l + 1$ for some $l \geq 1$. Following Lemma 4.3, g has a prefix $c \in K_1 * K_2$ such that $c^{-1}gc$ is cyclically reduced and $|c| \leq l$. There are two subcases to distinguish: $l' = l$ or $l' < l$.

If $l' = l$ then $c^{-1}gc = k_{l+1}r$ belongs to one of the factors: without loss of generality let us assume that $k_{l+1} \in K_1$ and consequently $c^{-1}gc \in G_1$.

Denote $k = k_{l+1} k_{l+1}^r \dots k_{l+1}^{r^{n-1}} \in K_1$. Then we have

$$\begin{aligned} g^n &= cc^{-1}g^n cc^{-1} = c(c^{-1}gc)c^{-1} \\ &= (k_1 \dots k_l)(k_{l+1}r)^n (k_1 \dots k_l)^{-1} \\ &= (k_1 \dots k_l)(kr^n)(k_1 \dots k_l)^{-1} \\ &= k_1 \dots k_l k (k_l^{-1})^{r^n} \dots (k_1^{-1})^{r^n}. \end{aligned}$$

If $k = e$ then $(c^{-1}gc)^n = r^n \in R$, i.e. $\langle c^{-1}gc \rangle \cap R \neq \{e\}$, which is a contradiction with the assumptions as $c^{-1}gc \notin R$. It follows that k_l and k belong to different factors, similarly for $(k_l^{-1})^{r^n}$ and k. It follows that $g^n = k_1 \dots k_l \overline{k} (k_l^{-1})^{r^n} \dots (k_1^{-1})^{r^n} r^n$ is the reduced expression for g^n and therefore g^n is not cyclically reduced as k_1 and $(k_1^{-1})^{r^n}$ belong to the same factor.

If $l' < l$, then the expression

$$c^{-1}gc = k_{l'+1} \dots k_{2l+1-l'} (k_{2l+2-l'} k_{l'}^r) r$$

is reduced, as can be seen in Equation (2). Let $w := k_{l'+1} \dots k_{2l-l'-1} (k_{2l-l'} k_{l'}^r)$ be an element of $K_1 * K_2$. It follows that

$$g^n = c(c^{-1}gc)^n c^{-1} = c(kr)^n c^{-1} = cww^r \dots w^{r^{n-1}} (c^{-1})^{r^n} r^n. \tag{4}$$

Note that the last letter of c and the fist letter of w belong to different factors, and the same is true for the first and last letter of w.

It follows that, up to replacing all occurrences of w with its expansion in terms of the elements k_i, the expression for g^n given in Equation (4) is reduced. The last letter of $w^{r^{n-1}}$ is $k_{2l+2-l'}^{r^{n-1}} k_{l'}^{r^n}$ and the first letter of $(c^{-1})^{r^n}$ is $(k_l^{-1})^{r^n}$. Multiplying those two we get $k_{2l-l'}^{r^{n-1}}$. It then follows that g^n is not cyclically reduced.

The last part of the statement follows from the reduced expression of Equation (3). □

We spell out the following fact, which was just proved in Lemma 4.4:

Corollary 4.5. *Let the groups G_i and G be as in Lemma 4.4, and $g = k_1 \dots k_l r$ be the reduced expression for the cyclically reduced element $g \in G$, with $l = |g|_* > 1$. Then*

$$g^n = k_1 \dots k_l k_1^r \dots k_l^r \dots k_1^{r^{n-1}} \dots k_l^{r^{n-1}} r^n$$

is the reduced expression of the element g^n, for all $n \geqslant 1$.

5. Separating cyclic subgroups of graph products in the profinite topology

The following result is proved by Bobrovskii and Sokolov in [3].

Theorem 5.1. *Let $G = G_1 *_R G_2$ be an amalgam over a common retract, let $g \in G$ be arbitrary, and suppose that G_1 and G_2 are residually finite. Then $\langle g \rangle$ is not separable in G if and only if g is conjugate to some $g_i \in G_i$, where $i \in \{1, 2\}$, and $\langle g_i \rangle$ is not separable in G_i.*

We will use the following lemma to shorten our proofs.

Lemma 5.2. *Let $G = \Gamma\mathcal{G}$ be a graph product of residually finite groups and let $g \in G$ be arbitrary. The cyclic subgroup $\langle g \rangle \leq G$ is separable in G if and only if it is separable in G_S, where $S = \mathrm{supp}(g)$. Furthermore, $\langle g \rangle$ is separable in G if and only if $\langle g' \rangle$ is separable in G for some (and hence for all) $g' \in g^G$.*

Proof. Graph products of residually finite groups are residually finite by [8, Corollary 5.4], hence G is residually finite. Since G_S is a retract of G, its profinite topology $\mathcal{PT}(G_S)$ is a restriction of $\mathcal{PT}(G)$ by Lemma 2.5. Therefore $\langle g \rangle$ is separable in G if and only if it is separable in G_S.

Now let $\phi \in \mathrm{Inn}(g)$ be an inner automorphism of G and let $\phi(g) = g'$. Clearly, $\langle g' \rangle = \langle \phi(g) \rangle = \phi(\langle g \rangle)$. Lemma 2.1 implies that ϕ is a homeomorphism of $\mathcal{PT}(G)$, hence $\langle g \rangle$ is separable in G if and only if $\langle g' \rangle$ is separable in G. □

Lemma 5.3. *Let $G = \Gamma\mathcal{G}$ be a graph product of residually finite groups and let $g \in G$ be a cyclically reduced element such that the full subgraph Γ_S contains a separating subset, where $S = \mathrm{supp}(g)$. Then the cyclic subgroup $\langle g \rangle \leq G$ is separable in G.*

Proof. Using Lemma 5.2, we may assume that $S = V\Gamma$ and subsequently $\Gamma = \Gamma_S$. As Γ is not complete, there exist a pair of vertices $u, v \in V\Gamma$ and a separating set $C \subseteq V\Gamma$ such that u, v lie in distinct connected components of $\Gamma \setminus C$, say $\Gamma_{A'}$ and $\Gamma_{B'}$, for $A', B' \subseteq V\Gamma \setminus C$. Without loss of generality we may assume that $V\Gamma = A' \cup B' \cup C$. Set $A = A' \cup C$ and $B = B' \cup C$.

As mentioned in Lemma 4.2, G splits as an amalgam over a common retract $G = G_A *_{G_C} G_B$. By Lemma 4.1, the element g can be written as

$$g = a_1 b_1 \dots a_n b_n r,$$

for some uniquely given $a_1, \dots a_n \in \ker(\rho_A)$, $b_1, \dots b_n \in \ker(\rho_B)$ and $r \in G_C$, where $\rho_A \colon G_A \to G_C$ and $\rho_B \colon G_B \to G_C$ are the canonical retractions.

As the element g is cyclically reduced and $\operatorname{supp}(g) = V\Gamma$, g cannot be conjugated to an element in any of the two groups $G_{V\Gamma\setminus\{v\}}$ or $G_{\operatorname{link}(v)} \times G_v$. By Theorem 5.1, it must be that $\langle g \rangle$ is separable in G. $\square$

Combining the lemma above with Lemma 5.2, we immediately get the following.

Corollary 5.4. *Let Γ be a graph, $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be a family of residually finite groups and $G = \Gamma\mathcal{G}$ be the corresponding graph product. Suppose that $g \in G$ is an arbitrary element such that Γ_S contains a separating subset, where $S = \operatorname{esupp}(g)$. Then the cyclic subgroup $\langle g \rangle \leq G$ is separable in G.*

Lemma 5.5. *Let G be a CSS group and let $C \leq G$ be an infinite cyclic subgroup. Then $\mathcal{PT}(C)$ is a restriction of $\mathcal{PT}(G)$.*

Proof. Let $N \leq C$ be open. As C is cyclic, by necessity, N is cyclic as well. By cyclic subgroup separability of G, the subgroup N is closed in G. Using Lemma 2.4 we get the result. $\square$

The following lemma can be seen as an slight strengthening of [5, Proposition 4.1], where it is shown that the direct product of quasi-potent CSS groups is again CSS. Using a slightly more topological approach, we show that quasi-potency is not necessary. The idea of using restrictions of profinite topologies was suggested to the authors by Ashot Minasyan, a suggestion for which we are very grateful.

Lemma 5.6. *The class of CSS groups is closed under forming direct products.*

Proof. Let G_1, G_2 be CSS groups and let $C \leq G_1 \times G_2$ be cyclic. Let $g = (g_1, g_2) \in G_1 \times G_2$ be a generator of C. Set $C_1 = \langle g_1 \rangle \leq G_1$ and $C_2 = \langle g_2 \rangle \leq G_2$. Using Lemma 5.5 we see that $\mathcal{PT}(C_1)$ is a restriction of $\mathcal{PT}(G_1)$ and pro-$\mathcal{C}(C_2)$ is a restriction of $\mathcal{PT}(G_2)$. It follows by Lemma 2.6 that $\mathcal{PT}(C_1 \times C_2)$ is a restriction of $\mathcal{PT}(G_1 \times G_2)$. Notice that $C_1 \times C_2$ is finitely generated abelian, hence it is LERF. This means that C is closed in $C_1 \times C_2$ and hence C is closed in $G_1 \times G_2$. $\square$

We are now ready to prove Theorem A:

Theorem A. The class of CSS groups is closed under forming graph products.

Proof. Let $G = \Gamma\mathcal{G}$ be a graph product of CSS groups and let $g \in G$ be arbitrary. Following Lemma 5.2, without loss of generality we may assume that g is cyclically reduced and that $\operatorname{supp}(g) = V\Gamma$.

If Γ is a complete graph, then $G = \prod_{v \in V\Gamma} G_v$ is a direct product of CSS groups, and thus it is CSS by Lemma 5.6.

If Γ is not complete, then it contains a separating subset, and in this case we can apply Corollary 5.4. $\square$

6. Unique roots

Definition 6.1 (**Unique roots**). Let G be a group, and $g \in G$ be an element. We say that an element $r \in G$ is a *root* of g if there is a positive integer $n \in \mathbb{N}$ such that $r^n = g$ in G. We say that $g \in G$ has *unique roots* if the equation $x^n = g$ has at most one solution for every $n \in \mathbb{N}$, i.e. for every $x, y \in G$ and every $n \in \mathbb{N}$ the equality $x^n = g = y^n$ implies $x = y$. A group G is said to have the *Unique Root property* if every $g \in G$ has unique roots.

We will use $\mathfrak{U}$ to denote the class of all groups with Unique Roots property.

As inverses are unique, replacing natural numbers by integers in the definition does not change the notion. Moreover, if a group has non-trivial torsion elements, then it does *not* have unique roots.

The aim of this section is to establish that the class $\mathfrak{U}$ is closed under taking graph products. We start with a fact that will be used in Proposition 6.4.

Lemma 6.2. *Let G be a group and let $R \leq G$ be a retract. If $G \in \mathfrak{U}$ then for every $g \in G \setminus R$ we have $\langle g \rangle \cap R = \{e\}$.*

Proof. Suppose that there is $g \in G \setminus R$ and a $n \in \mathbb{N}$ such that $g^n \in R \setminus \{e\}$. Let $\rho\colon G \to R$ be the retraction corresponding to R and set $\rho(g) = r \in R$. We see that $g^n = \rho(g^n) = r^n$. However $g \neq r$, contradicting the unique root property. $\square$

Lemma 6.3. *The class $\mathfrak{U}$ is closed under taking subgroups and direct products.*

Proof. Let G be a group with Unique Root property and suppose that $H \leq G$. Let $x, y \in H$ be arbitrary and suppose that $x^n = y^n$ for some $n \in \mathbb{N}$. As $x, y \in G$ and $G \in \mathfrak{U}$, we see that $x = y$.

For direct products, we prove the statement for a direct product of two groups. The argument applies to any number (finite or not) of direct factors. Let $G_1, G_2 \in \mathfrak{U}$, let $x = (x_1, x_2), y = (y_1, y_2) \in G_1 \times G_2$ be arbitrary elements and suppose that $x^n = y^n$ for some $n \in \mathbb{N}$. This means that $(x_1^n, x_2^n) = (y_1^n, y_2^n)$, i.e. $x_1^n = y_1^n$ in G_1 and $x_2^n = y_2^n$ in G_2. By unique roots, we conclude that $x_1 = y_1$ in G_1, and that $x_2 = y_2$ in G_2. Therefore $x = y$, and thus the direct product $G_1 \times G_2$ has the unique roots property. $\square$

In particular, any retract of a group with the unique root property also has the unique roots property.

In the following proposition we prove that unique roots is preserved under taking amalgamations along retracts, and in particular by free products.

Proposition 6.4. *The class $\mathfrak{U}$ is closed under taking amalgams over retracts.*

Proof. Let $G_1 = K_1 \rtimes R$, $G_2 = K_2 \rtimes R$ be groups in $\mathfrak{U}$, let $\rho_1 \colon G_1 \to R$ and $\rho_2 \colon G_2 \to R$ be the corresponding canonical retractions. Set

$$G = G_1 *_R G_2 \cong (K_1 * K_2) \rtimes R.$$

and let $\rho \colon G \to R$ be the natural extension of ρ_1, ρ_2 to G.

Let $x, y \in G$ be arbitrary elements such that $x^n = y^n$ for some $n \geq 2$. Let $r_x, r_y \in R$ and $k_x, k_y \in K_1 * K_2$ be the uniquely given elements such that $x = k_x r_x$ and $y = k_y r_y$.

As $\rho(x^n) = \rho(y^n)$, we have that $r_x^n = r_y^n$. The retract R has the Unique Root property by Lemma 6.3, and therefore we conclude that $r_x = r_y$, which we denote by r. We see that

$$x^n = k_x k_x^r \dots k_x^{r^{n-1}} r^n,$$
$$y^n = k_y k_y^r \dots k_y^{r^{n-1}} r^n.$$

As $x^n = y^n$, we obtain that

$$k_x k_x^r \dots k_x^{r^{n-1}} = k_y k_y^r \dots k_y^{r^{n-1}}, \tag{5}$$

and we denote this element by k.

Without loss of generality, we can suppose that x^n (and, consequently, also y^n) is cyclically reduced. Indeed, if this is not the case, by Lemma 4.3 there exists a prefix c of x^n such that $c^{-1}x^n c$ is cyclically reduced. Therefore, we can replace x^n and y^n with $c^{-1}x^n x$ and $c^{-1}y^n x$ and proceed considering these elements.

By Lemma 6.2, the groups G_1 and G_2 satisfy the hypotheses of Lemma 4.4. Therefore, applying it, we see that both x and y are cyclically reduced. Following Lemma 4.4, we see that $|k|_* = 1$ if and only if $|k_x|_* = 1$, if and only if $|k_y|_* = 1$. In this case, it follows that both elements k_x and k_y must belong to the same factor, which without loss of generality we assume to be K_1. This means that both $x, y \in G_1$, and from $x^n = y^n$ we conclude that $x = y$, as $G_1 \in \mathfrak{U}$.

Now suppose that $|k|_* > 1$. Therefore $|k_x|_*$ and $|k_y|_*$ are greater than one by Lemma 4.4, and moreover $|k| = n|k_x| = n|k_y|$. Suppose that $k_x = k_1 \dots k_l$ is a reduced expression for k_x in $K_1 * K_2$, and that $k_y = h_1 \dots h_m$ is a reduced expression for k_y. As we assumed x^n to be cyclically reduced, we conclude that both these expressions are cyclically reduced.

From Equation (5), k can be expressed as

$$\begin{aligned}(k_1 \dots k_l)(k_1^r \dots k_l^r) \dots (k_1^{r^{n-1}} \dots k_l^{r^{n-1}}) = \\ = (h_1 \dots h_m)(h_1^r \dots h_m^r) \dots (h_1^{r^{n-1}} \dots h_m^{r^{n-1}}).\end{aligned}$$

By the normal form theorem for free products (see [12, Theorem 4.1] or [11, Chapter IV, Theorem 1.2]) we obtain that $l = m$ and $k_i = h_i$ for

$i = 1, \dots, m$. Hence $k_x = k_y$, and therefore

$$x = k_x r = k_y r = y.$$

Thus $G \in \mathfrak{U}$. □

From the previous results, we can conclude:

Theorem 6.5. *The class $\mathfrak{U}$ is closed under taking graph products.*

Proof. Let $\Gamma = (V\Gamma, E\Gamma)$ be a graph, let $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be a family of groups in $\mathfrak{U}$, and let G denote the corresponding graph product.

We proceed by induction on the number of vertices of Γ. If $|V\Gamma| = 1$ then the statement holds trivially. Now let $n \in \mathbb{N}$ be given and suppose that the statement has been proved for all graphs Γ' with $|V\Gamma'| < n = |V\Gamma|$.

If the graph Γ contains a separating set $S \subset V\Gamma$, then by Lemma 4.2 the group G properly splits as an amalgam over a retract $G = G_1 *_{G_S} G_2$, for some proper full subgroups $G_1, G_2, G_S \leq G$. By induction hypothesis the groups G_1, G_2 have the Unique Root property, and therefor $G \in \mathfrak{U}$ by Proposition 6.4.

If G does not contain a separating set, then Γ is a complete graph. Therefore, G is the direct product of the vertex groups, $G = \prod_{v \in V\Gamma} G_v$, and then $G \in \mathfrak{U}$ by Lemma 6.3.

This induction also proves the statement for an infinite graph Γ: any equality $x^n = y^n$ can be seen in the full subgroup associated to the finite subgraph whose vertices are $\operatorname{esupp}(x) \cup \operatorname{esupp}(y)$. Therefore, a failure of Unique Roots in G would produce a contradiction with what we just proved. □

Since $\mathbb{Z} \in \mathfrak{U}$, we re-obtain the following corollary, originally proven in [15, Lemma 6.3].

Corollary 6.6. *Right-angled Artin groups satisfy the Unique Roots property.*

7. Primitive roots and p-isolation

From now, we will consider the class $\mathcal{C}$ to consist of all finite p-groups for some prime number p. Given a group G, we use $\mathcal{N}_p(G)$ to denote the set of all co-p-finite subgroups of G and we use pro-$p(G)$ to denote the pro-p topology on G. Also, for a subset $X \subseteq G$ we use the term p-separable or p-closed in G to denote that X is closed in the pro-p topology on G.

First, let us consider the following example.

Example 7.1. Let G be a an arbitrary infinite group, suppose that there is $g_0 \in G$ such that $\operatorname{ord}_G(g_0) = \infty$ and $g \in G$ such that $g_0 = g^q$ for some prime number q distinct from p. Note that $g \notin \langle g_0 \rangle$ and $\langle g_0 \rangle \leq \langle g \rangle$. Assume that $\pi \colon G \twoheadrightarrow Q$ is a surjective homomorphism onto some finite p-group Q. Then $\langle \pi(g) \rangle$ is a cyclic group of order p^e for some $e \in \mathbb{N}$. As $\pi(g_0) = \pi(g)^q$ and $\gcd(q, p^e) = 1$, we see that $\pi(g_0)$ generates $\langle \pi(g) \rangle$

and therefore $\pi(g) \in \langle \pi(g_0) \rangle = \pi(\langle g_0 \rangle)$. In particular, the cyclic subgroup $\langle g_0 \rangle \leq G$ is not closed in the pro-p topology on G.

This example motivates the following definition.

Definition 7.2 (**p-isolation**)**.** Let G be a group and let $H \leq G$. We say that H is *p-isolated* in G if for every $f \in G$ and every prime number q distinct from p the following holds:

$$f^q \in H \quad \Rightarrow \quad f \in H.$$

An element $g \in G$ is said to be p-isolated in G if the cyclic subgroup $\langle g \rangle$ is p-isolated in G.

The authors of [3] use the term *p'-isolated* for the same notion. To ease the notation, we decided to drop the $'$ as there is no chance of confusion.

Following Example 7.1, we see that being p-isolated is a necessary condition for a subgroup to be p-separable, hence it makes sense to consider p-separability only for p-isolated subgroups. However, it was shown in [1] that a non-abelian free group contains a finitely generated subgroup which is p-isolated but not p-separable for any prime number p.

Therefore, we pose the following definition:

Definition 7.3 (**p-cyclic subgroup separability**)**.** A group G is *p-cyclic subgroup separable* (p-CSS) if every p-isolated cyclic subgroup of G is p-separable in G.

The aim of this section is to give a useful description of p-isolated cyclic subgroups of groups.

Lemma 7.4. *Let G be a group and suppose that $g \in G$ is of infinite order. The cyclic group $\langle g \rangle \leq G$ is p-isolated if and only if for every $n \in \mathbb{N}$ coprime to p and every $f \in G$*

$$f^n \in \langle g \rangle \quad \Rightarrow \quad f \in \langle g \rangle.$$

Proof. Only one implication is non trivial, so let $f \in G$ be arbitrary and suppose that $f^n \in \langle g \rangle$ for some n coprime to p. Let $n = p_1^{e_1} \dots p_m^{e_m}$ be the prime factorisation of n. We will proceed by induction on $N = e_1 + \dots + e_m$.

If $N = 1$ then n is a prime and the statement holds. Suppose that the statement has been proved for all n' whose sum of exponents in the prime decomposition is less than N. Set $f' = f^{p_1}$ and $n' = n/p_1$. Note that n' is coprime with p. As $(f')^{n'} = f^n \in \langle g \rangle$, we have that $f' \in \langle g \rangle$ by induction hypothesis. Moreover $f' = f^{p_1}$, and therefore $f \in \langle g \rangle$. $\square$

Informally speaking, a subgroup is p-isolated if it is closed under taking "n-th roots" for n coprime with p. This informal observation motivates the rest of this section.

Definition 7.5 (**Primitive roots, and primitive logarithms**)**.** Let G be a group. As defined in the previous section, an element $r \in G$ is a root of $g \in G$ if there is a positive integer $k \in \mathbb{N}$ such that $r^k = g$ in G. We say that r is a *primitive root* of g in G if such k is maximal possible:

$$k = \max\{n \in \mathbb{N} \mid \exists r \in G\colon r^n = g\}.$$

We use $\sqrt[G]{g}$ to denote the set of all primitive roots of g in G. When the group G has unique roots, we slightly abuse notation and use $\sqrt[G]{g}$ to denote the primitive root of g in G.

If r is a primitive root of g in G with corresponding exponent $k \in \mathbb{N}$ so that $r^k = g$, then we say that k is the *primitive logarithm* of g in G, and we denote this as $k = \operatorname{plog}_G(g)$.

If $g \in G$ has finite order n, then $\sqrt[G]{g}$ is empty because $g^{kn} = e_G$ for all $k \in \mathbb{N}$, and therefore there is no maximal.

However, this is not the only case when an element $g \in G$ might not have a primitive root. Indeed, consider the Baumslag-Solitar group

$$G = \mathrm{BS}(1,2) = \langle a, t \| tat^{-1} = a^2 \rangle.$$

From the relation of G one deduces that $(t^{-n}at^n)^{2^n} = a$ for every $n \in \mathbb{N}$, and therefore the element a has no primitive roots: $\sqrt[G]{a} = \emptyset$.

Remark 7.6. Let G be a group and let $g \in G$ be an element with primitive root. For any $c \in G$ we have that

$$\operatorname{plog}_G(cgc^{-1}) = \operatorname{plog}_G(g), \qquad \sqrt[G]{cgc^{-1}} = c\,(\sqrt[G]{g})\,c^{-1}.$$

Consider a group G given by the presentation

$$\langle x, y \| x^p = y^q \rangle,$$

where $p < q$ are distinct primes. Then the element x is its own (unique) primitive root, that is $\sqrt[G]{x} = \{x\}$, but $\sqrt[G]{x^p} = y$ and $\operatorname{plog}_G(x^p) = q$.

This motivates the following definition.

Definition 7.7 (**Primitive stability).** We say that an element $g \in G$ is *primitively stable* in G if $\sqrt[G]{g}$ is defined and $\operatorname{plog}_G(g^n) = n \cdot \operatorname{plog}_G(g)$ for all $n \in \mathbb{N}$. We say that a group G is *primitively stable* if every $g \in G \setminus \{e\}$ is primitively stable.

Note that primitively stable groups are necessarily torsion-free. We denote by $\mathfrak{U}_{\mathrm{ps}}$ the class of primitively stable groups with unique roots.

The following will not be used during the text, but it provides a nice characterisation for primitively stable elements and provides a comparison to Proposition 9.2.

Lemma 7.8. *Let G be a group. For an element $g \in G$, we have that $\operatorname{plog}_G(g^n) = n \cdot \operatorname{plog}_G(g)$ if and only if $\sqrt[G]{g} \subseteq \sqrt[G]{g^n}$.*

Proof. Suppose that $\operatorname{plog}_G(g^n) = n \cdot \operatorname{plog}_G(g)$, and let $r \in \sqrt[G]{g}$, so that $r^{plog_G(g)} = g$. By taking powers, we obtain that $r^{n \cdot plog_G(g)} = g^n$, and the hypothesis implies that $r^{plog_G(g^n)} = g^n$. This, by definition, means that $r \in \sqrt[G]{g^n}$. Therefore $\sqrt[G]{g} \subseteq \sqrt[G]{g^n}$.

Suppose now that $\sqrt[G]{g} \subseteq \sqrt[G]{g^n}$, and let $r \in \sqrt[G]{g}$, so that $r^{plog_G(g)} = g$. Again by taking the n-th power, we obtain that $r^{n \cdot plog_G(g)} = g^n$. By assumption $r \in \sqrt[G]{g^n}$, and therefore $r^{plog_G(g^n)} = g^n$, so that

$$r^{plog_G(g^n)} = r^{n \cdot plog_G(g)} = g^n.$$

As r is a primitive root for g^n, it must follow that $plog_G(g^n) = n \cdot plog_G(g)$. □

Lemma 7.9. *Let $G \in \mathfrak{U}_{\mathrm{ps}}$, and let $x, y \in G$ be such that $x^m = y^n$ for some $m, n \in \mathbb{Z}$. Then there is $r \in G$ such that $x, y \in \langle r \rangle$. In particular, $\sqrt[G]{x} = \sqrt[G]{y} = \{r\}$.*

Proof. Let $r_x = \sqrt[G]{x}$ and $k_x = \mathrm{plog}_G(x)$, so that $r_x^{k_x} = x$, and similarly $r_y = \sqrt[G]{y}$, $k_y = \mathrm{plog}_G(y)$. From primitive stability we obtain that $\mathrm{plog}_G(x^m) = m \cdot \mathrm{plog}_G(x) = mk_x$, and analogously that $\mathrm{plog}_G(y^n) = nk_y$.

As $x^m = y^n$, by unique roots we have that $\mathrm{plog}_G(x^m) = \mathrm{plog}_G(y^n)$, that is $mk_x = nk_y$, which we denote by k. Therefore $r_x^k = r_y^k$, and we conclude that $r_x = r_y$, as $G \in \mathfrak{U}$. Thus $x, y \in \langle r \rangle$, where r denotes $r_x = r_y$. □

Lemma 7.10. *Let $G \in \mathfrak{U}_{\mathrm{ps}}$ and $g \in G$ be arbitrary. The subgroup $\langle g \rangle$ is p-isolated in G if and only if $\mathrm{plog}_G(g)$ is a power of p.*

Proof. Suppose that $\mathrm{plog}_G(g)$ is not a power of p, i.e. $\mathrm{plog}_G(g) = mp^e$ for some m coprime with p. For $r = \sqrt[G]{g}$, we have that $r^{p^e} \notin \langle g \rangle$, but $(r^{p^e})^m = g \in \langle g \rangle$. Hence $\langle g \rangle$ is not p-isolated.

Assume now that $\mathrm{plog}_G(g) = p^e$. Let $f \in G$ and suppose that $f^q \in \langle g \rangle$ for some prime q distinct from p, so that $f^q = g^k$ for some $k \in \mathbb{Z}$. By Lemma 7.9 we see that $\sqrt[G]{f} = \sqrt[G]{g} = \{r\}$.

Set $n = \mathrm{plog}_G(f)$, so that $r^n = f$. We have that

$$r^{kp^e} = g^k = f^q = r^{nq},$$

and hence $kp^e = nq$. As q is a prime distinct from p, it must divide k, thus $n = k/q \cdot p^e$ with k/q a natural number. This means that

$$f = r^n = \left(r^{p^e}\right)^{k/q} = g^{k/q} \in \langle g \rangle,$$

and therefore we proved that $\langle g \rangle$ is p-isolated in G. □

Notice that, in the previous lemma, we used the fact that G had the Unique Roots property just for one implication.

For the next fact, let us remember that if G is a free abelian group freely generated by $a_1, \ldots, a_n$, then any element $g \in G$ can be written as $g = (a_1^{k_1}, \ldots, a_n^{k_n})$ for uniquely given $k_1, \ldots, k_n \in \mathbb{Z}$.

Lemma 7.11. *Let G be a finitely generated free abelian group with a free base $\{a_1, \ldots, a_n\}$. An element $g = (a_1^{k_1}, \ldots, a_n^{k_n})$ is p-isolated if and only if $\gcd(k_1, \ldots, k_n)$ is a power of p.*

Proof. Let $f = (a_1^{l_1}, \dots, a_n^{l_1}) \in G$ be a non-trivial element, let q be a prime distinct from p, and suppose that $f^q \in \langle g \rangle$. This means that there is a $z \in \mathbb{Z} \setminus \{0\}$ such that

$$k_1 z = l_1 q, \qquad \dots \qquad k_n z = l_n q. \tag{6}$$

Note that $k_i = 0$ if and only if $l_i = 0$. Moreover, q divides $k_i z$ for $1 \leq i \leq n$. As q is a prime, then q either divides z or it divides every k_i.

Suppose that $z = z'q$ for some $z' \in \mathbb{Z}$. By Equation (6), we conclude that $k_i z' q = l_i q$ for all i, that is $k_i z' = l_i$. Thus $f \in \langle g \rangle$, as $f = g^{z'}$.

If q does not divide z, then q must divide every k_i. It follows that q divides $\gcd(k_1, \dots, k_n)$, so the greatest common divisor cannot be a power of p. □

Primitive roots and primitive logarithms do not necessarily behave in a stable manner with respect to subgroups. Consider an infinite cyclic group $G = \langle g \rangle$ and let $K = \langle g^k \rangle \leq G$, for some $k \geq 2$. Then $\sqrt[K]{g^{kl}} = g^k$ and $\mathrm{plog}_K(g^{kl}) = l$, whereas $\sqrt[G]{g^{kl}} = g$ and $\mathrm{plog}_G(g^{kl}) = kl$.

Lemma 7.12. *Let $G \in \mathfrak{U}_{\mathrm{ps}}$ and $H \leq G$. For every $h \in H$ we have that*

$$\sqrt[H]{h} = (\sqrt[G]{h})^a, \qquad \mathrm{plog}_H(h) = \frac{\mathrm{plog}_G(h)}{a},$$

where $a = |\langle \sqrt[G]{h} \rangle \colon (\langle \sqrt[G]{h} \rangle \cap H)|$.

Furthermore H is primitively stable, and therefore the class $\mathfrak{U}_{\mathrm{ps}}$ is closed under taking subgroups.

Proof. Let $r = \sqrt[G]{h}$. By primitive stability we see that a divides $\mathrm{plog}_G(h)$, denote $\mathrm{plog}_G(h) = ak$.

Suppose that there is $\tilde{r} \in H$ such that $\tilde{r}^s = h$ for some $s \in \mathbb{N}$. Using Lemma 7.9 we see that $\sqrt[G]{\tilde{r}} = r = \sqrt[G]{r}$, hence $\tilde{r} \in \langle r \rangle \cap H$. As $a = \min\{n \in \mathbb{N} \mid r^n \in H\}$, it follows that $\tilde{r} \in \langle r^a \rangle$. We see that $k = \mathrm{plog}_H(h)$. By unique roots, it follows that $r^a = \sqrt[H]{h}$.

Now consider h^n for some $n \in \mathbb{N}$. By primitive stability in G we see that

$$\left| \left\langle \sqrt[G]{h^n} \right\rangle : \left\langle \sqrt[G]{h^n} \right\rangle \cap H \right| = \left| \left\langle \sqrt[G]{h} \right\rangle : \left\langle \sqrt[G]{h} \right\rangle \cap H \right|.$$

By the first part of the statement $\mathrm{plog}_H(h^n) = \mathrm{plog}_G(h^n)/a$. By primitive stability in G we see that

$$\mathrm{plog}_H(h^n) = \frac{\mathrm{plog}_G(h^n)}{a} = n \frac{\mathrm{plog}_G(h)}{a} = n \, \mathrm{plog}_H(h),$$

hence H is primitively stable. □

However, primitive roots and primitive logarithms are stable with respect to retracts.

Lemma 7.13. *Let $G \in \mathfrak{U}$, suppose that $R \leq G$ is a retract such that R is primitively stable, and let $\rho \colon G \to R$ denote the corresponding retraction.*

Then $\operatorname{plog}_G(g)$ *divides* $\operatorname{plog}_R(\rho(g))$ *for all* $g \in G$*. Moreover, if* $g \in R$*, then* $\sqrt[G]{g} = \sqrt[R]{g}$ *and* $\operatorname{plog}_R(g) = \operatorname{plog}_G(g)$*.*

Proof. Let $r = \sqrt[G]{g}$ and $k = \operatorname{plog}_G(g)$, so that $r^k = g$ and, therefore, $\rho(r)^k = \rho(g)$. Let $\tilde{r} = \sqrt[R]{\rho(g)}$ and $\tilde{k} = \operatorname{plog}_R(\rho(g))$. We see that

$$\tilde{r}^{\tilde{k}} = \rho(g) = \rho(r)^k,$$

and therefore by primitive stability we obtain that

$$\operatorname{plog}_R(\rho(g)) = \operatorname{plog}_R(\rho(r)^k) = k \cdot \operatorname{plog}_R(\rho(r)).$$

As $k = \operatorname{plog}_G(g)$, we showed that $\operatorname{plog}_G(g)$ divides $= \operatorname{plog}_R(\rho(g))$

Suppose that $g \in R$, so that $\rho(g) = g$. Then $\rho(r)^k = \rho(g) = g = r^k$ and by the Unique Root property we see that $\rho(r) = r$, i.e. $\sqrt[G]{g} \in R$. By Lemma 7.9 we see that $r = \sqrt[R]{g}$ and $k = \operatorname{plog}_R(g)$. □

Definition 7.14 (**p-inseparability**)**.** Let G be a group and $f, g \in G$. We say that the pair $(f, \langle g \rangle)$ is *p-inseparable* if $\pi_N(f) \in \langle \pi_N(g) \rangle$ for every $N \in \mathcal{N}_p(G)$.

The following lemma shows that for groups in $\mathfrak{U}_{\mathrm{ps}}$ which are p-CSS, the only inseparable pairs arise from the counterexamples to p-isolation (see Example 7.1).

Lemma 7.15. *Let* $G \in \mathfrak{U}_{\mathrm{ps}}$ *be a p-CSS group and let* $f, g \in G$ *be such that* $f \notin \langle g \rangle$ *and the pair* $(f, \langle g \rangle)$ *is p-inseparable. Then* $\sqrt[G]{f} = \sqrt[G]{g}$ *and* $f^k \in \langle g \rangle$ *for some* $k \in \mathbb{Z}$ *coprime to* p*.*

Proof. As G is p-CSS, we see that g cannot be p-isolated. Set $r = \sqrt[G]{g}$. As g is not p-isolated, from Lemma 7.10 we see that $\operatorname{plog}_G(g) = kp^e$, where p does not divide $k > 1$.

Set $g' = r^{p^e}$. Notice that $\operatorname{plog}_G(g') = p^e$, hence g' is p-isolated in G by Lemma 7.10 and, in particular, $\langle g' \rangle$ is p-closed in G.

If $f \notin \langle g' \rangle$, then there is $N \in \mathcal{N}_p$ such that $\pi_N(f) \notin \langle \pi_N(g') \rangle$. As $\langle g \rangle \leq \langle g' \rangle$, it would follow that $\pi_N(f) \notin \langle \pi_N(g) \rangle$, contradicting the fact that $(f, \langle g \rangle)$ is p-inseparable.

Therefore $f \in \langle g' \rangle$ and $f^k \in \langle g \rangle$. Moreover, by Lemma 7.9 we conclude that $\sqrt[G]{f} = \sqrt[G]{g}$. □

8. Primitive stability in graph products

The following lemma is a direct consequence of Lemma 7.13 together with Definition 7.5 and Definition 7.7.

Lemma 8.1. *Let* $G_1, G_2 \in \mathfrak{U}_{\mathrm{ps}}$ *and* $g = (g_1, g_2) \in G_1 \times G_2 = G$ *be arbitrary. Then*

$$\operatorname{plog}_G(g) = \gcd\left(\operatorname{plog}_{G_1}(g_1), \operatorname{plog}_{G_2}(g_2)\right).$$

and

$$\sqrt[G]{g} = \left(\sqrt[G_1]{g_1}^{\frac{\operatorname{plog}_{G_1}(g_1)}{\operatorname{plog}_G(g)}}, \sqrt[G_2]{g_2}^{\frac{\operatorname{plog}_{G_2}(g_2)}{\operatorname{plog}_G(g)}} \right). \tag{7}$$

Proof. The groups G_1 and G_2 are both retracts of the direct product $G_1 \times G_2$. Therefore, from Lemma 7.13 it follows that $\mathrm{plog}_G(g)$ divides both $\mathrm{plog}_{G_1}(g_1)$ and $\mathrm{plog}_{G_2}(g_2)$, and hence their greatest common divisor. As the primitive logarithm is defined to be the exponent of the primitive root, it must be that $\mathrm{plog}_G(g)$ is that greatest common divisor.

The equality of Equation (7) follows from the previous argument. □

Remark 8.2. Notice that primitive stability is an essential hypothesis for Lemma 8.1. Indeed, consider the groups

$$G_n := \langle g_n, r_{n,2}, \ldots, r_{n,n} \parallel g_n = r_{n,i}^i \; \forall i = 2, \ldots, n \rangle.$$

for $n \geqslant 2$. These groups are not primitively stable, and $G_n \times G_{n-1}$ does not satisfy the conclusion of Lemma 8.1, because

$$\mathrm{plog}_{G_n \times G_{n-1}}(g_n, g_{n-1}) = n - 1$$

does not divide the great common divisor of $n-1$ and n.

Lemma 8.3. *The class $\mathfrak{U}_{\mathrm{ps}}$ is closed under direct products.*

Proof. Let $G_1, G_2 \in \mathfrak{U}_{\mathrm{ps}}$, consider $G = G_1 \times G_2$ and let $n \in \mathbb{N}$, $g = (g_1, g_2) \in G$ be arbitrary. By Lemma 6.3 we see that G has Unique Roots property, that is $G \in \mathfrak{U}$. From Lemma 8.1, and exploiting primitive stability for the second equality, we get that

$$\begin{aligned} \mathrm{plog}_G(g^n) &= \gcd\left(\mathrm{plog}_{G_1}(g_1^n), \mathrm{plog}_{G_2}(g_2^n)\right) \\ &= \gcd\left(n\,\mathrm{plog}_{G_1}(g_1), n\,\mathrm{plog}_{G_2}(g_2)\right) \\ &= n\gcd\left(\mathrm{plog}_{G_1}(g_1), \mathrm{plog}_{G_2}(g_2)\right) = n\,\mathrm{plog}_G(g). \end{aligned}$$

Therefore G is also primitively stable. □

Lemma 8.4. *Let $G_1 = K_1 \rtimes R$, $G_2 = K_2 \rtimes R$ be groups in $\mathfrak{U}$ and suppose that R is primitively stable. Let $x, y \in G = G_1 *_R G_2$ be cyclically reduced elements with $|x|_*, |y|_* > 1$, and suppose that $x^m = y^n$ for some $m, n \in \mathbb{N}$. Then there are $z \in G$ and $a, b \in \mathbb{N}$ such that $z^a = x$ and $z^b = y$.*

Proof. Let $\rho\colon G \to R$ be the canonical retraction. As $\rho(x)^m = \rho(y)^m$, by Lemma 7.9 we see that $\rho(x)$ and $\rho(y)$ have a common primitive root in R, denoted by r_0. Following Proposition 6.4 we see that $G \in \mathfrak{U}$, therefore by Lemma 7.13

$$\mathrm{plog}_R(\rho(x)) = \mathrm{plog}_G(\rho(x)), \qquad \sqrt[R]{\rho(x)} = \sqrt[G]{\rho(x)},$$

and the analogous equalities hold also for $\rho(y)$. Set

$$e = \gcd\bigl(\mathrm{plog}_R(\rho(x)), \mathrm{plog}_R(\rho(y))\bigr)$$

and

$$a = \frac{\mathrm{plog}_R(\rho(x))}{e}, \qquad b = \frac{\mathrm{plog}_R(\rho(y))}{e},$$

so that $\gcd(a,b)=1$. Denoting $r := r_0^c$, we have that $r_0^{ac} = r^a = \rho(x)$, and $r^b = \rho(y)$, and therefore $ma = nb$.

As both x, y are cyclically reduced, and $|x|_*, |y|_* > 1$, we see that $|x^m|_* = m|x|_*$ and $|y^n|_* = n|y|_*$ by Lemma 4.4. Thus $m|x|_* = n|y|_*$, and in fact

$$|x|_* = al, \qquad |y|_* = bl,$$

where $l = \gcd(|x|_*, |y|_*)$. Notice that $2 \mid l$, and therefore $l \neq 1$.

Let $x = k_1 \dots k_{al} r^a$ be the reduced expression for x. By Corollary 4.5, we have that

$$y^n = x^m = k_1 \dots k_{al} k_1^{r^a} \dots k_{al}^{r^a} \dots k_1^{r^{a(m-1)}} \dots k_{al}^{r^{a(m-1)}} r^{am}, \tag{8}$$

is the (unique) reduced expression for $x^m = y^n$, where $k_1, \dots, k_{al} \in K_1 \cup K_2$. Progressively rename the aml elements of $K_1 \cup K_2$ in the right-hand side of Equation (8) as $k_1, \dots, k_{aml}$, so that $x^m = k_1 \dots k_{aml} r^{am}$.

By assumption we have $k_{i+al} = k_i^{r^a}$ for $i = 1, \dots, (m-1)al$ and moreover $k_{i+bl} = k_i^{r^b}$ for $i = 1, \dots, (n-1)bl$. From these identities we get that

$$k_{i+c_1al+c_2bl} = k_i^{r^{c_1a+c_2b}}$$

for a suitable choice of $c_1, c_2 \in \mathbb{Z}$. Suppose that $c_1, c_2 \in \mathbb{Z}$ are some Bézout's coefficients for $\operatorname{plog}_R(\rho(x))$, $\operatorname{plog}_R(\rho(y))$, i.e.

$$e = \gcd\big(\operatorname{plog}_R(\rho(x)), \operatorname{plog}_R(\rho(y))\big) = c_1 \operatorname{plog}_R(\rho(x)) + c_2 \operatorname{plog}_R(\rho(y)).$$

It then follows that c_1, c_2 are Bézout's coefficients for a, b, i.e. $c_1a + c_2b = 1$. In particular, we see that $c_1al + c_2bl = l$ and, consequently, we see that

$$k_{i+l} = k_{i+c_1al+c_2bl} = k_i^{r^{c_1a+c_2b}} = k_i^r$$

for all $i = 1, \dots, (am-1)l$.

If we set $z = k_1 \dots k_l r \in G$, we see that $z^a = x$ and $z^b = y$. $\square$

Proposition 8.5. *The class $\mathfrak{U}_{\mathrm{ps}}$ is closed under taking amalgams over retracts.*

Proof. Let $G_1 = K_1 \rtimes R$, $G_2 = K_2 \rtimes R$ be groups in $\mathfrak{U}_{\mathrm{ps}}$, and consider $G = G_1 *_R G_2 \simeq (K_1 * K_2) \rtimes R$, their amalgam along the common retract R.

By Proposition 6.4 we have that $G \in \mathfrak{U}$. To prove that G is also primitively stable, consider an element $g \in G$ and $n \in \mathbb{N}$. Following Remark 7.6, without loss of generality we may assume that g is cyclically reduced. There are two distinct cases to consider: either g belongs to one of the factors, or not.

Suppose that g belongs to one of the factors, assume $g \in G_1$. The group G_1 is a retract of G, with the retraction map ρ_1 defined on the generators of G in the following manner:

$$\rho_1(g) = \begin{cases} g & \text{if } g \in R, \\ g & \text{if } g \in K_1, \\ 1 & \text{if } g \in K_2. \end{cases}$$

From Lemma 7.13

$$\operatorname{plog}_G(g^n) = \operatorname{plog}_{G_1}(g^n) = n \operatorname{plog}_{G_1}(g) = n \operatorname{plog}_G(g),$$

and therefore the element g is primitely stable.

For the remaining case, suppose that g does not belong to a factor. Therefore $|g|_* > 1$. Let

$$x = \sqrt[G]{g}, \qquad m = \operatorname{plog}_G(g)$$

and

$$y = \sqrt[G]{g^n}, \qquad e = \operatorname{plog}_G(g^n).$$

As g is cyclically reduced, by Lemma 4.4 we see that the three elements x, $x^{mn} = g^n = y^e$, and y are cyclically reduced. By definition of primitive logarithm we have that

$$e = \operatorname{plog}_G(g^n) \geqslant \operatorname{plog}_{G_1}(g^n) = mn.$$

By Lemma 8.4, there is an element $z \in G$ and natural numbers $a, b \in \mathbb{N}$ such that $z^a = x$ and $z^b = y$. Therefore $g^n = z^{be}$, which is a contradiction with the maximality of $\operatorname{plog}_G(g^n)$ unless $b = 1$. Hence $z = y$, by unique roots. Similarly $z^a = x$ and, consequently, $g = z^{am}$. Again, this implies that $a = 1$, and therefore $x = y$, that is $\sqrt[G]{g^n} = \sqrt[G]{g}$ and $\operatorname{plog}_G(g^n) = n \operatorname{plog}_G(g)$. □

The proof of the following statement is analogous to the proof of Theorem 6.5, where instead of Lemma 6.3 and Proposition 6.4 one would use Lemma 8.3 and Proposition 8.5, respectively.

Theorem 8.6. *The class $\mathfrak{U}_{\mathrm{ps}}$ is closed under taking graph products.*

As an immediate corollary:

Corollary 8.7. *Right-angled Artin groups belong to $\mathfrak{U}_{\mathrm{ps}}$*

9. Examples of primitively stable groups with Unique Roots property

Let G be a group and $g \in G$. We define *the radical of g in G* as the subset

$$\operatorname{Rad}_G(g) := \{r \in G \mid r^a \in \langle g \rangle \text{ for some } a \in \mathbb{Z} \setminus \{0\}\}.$$

Notice that the set $\operatorname{Rad}_G(g)$ does not have to be a subgroup of G. Indeed, let $G = \langle x, y \parallel x^2 = y^2 \rangle$ be the fundamental group of the Klein bottle. Then $x, y \in \operatorname{Rad}_G(x^2)$, but $xy \notin \operatorname{Rad}_G(g)$.

Nevertheless, in Proposition 9.2 we prove that $\operatorname{Rad}_G(g)$ is a cyclic subgroup for any non-trivial element g of a group $G \in \mathfrak{U}_{\mathrm{ps}}$.

Lemma 9.1. *Let G be a group and let $g_1, g_2 \in G \setminus \{e\}$ be arbitrary. If $\operatorname{Rad}_G(g_1) \cap \operatorname{Rad}_G(g_2) \neq \{e\}$ then $\operatorname{Rad}_G(g_1) = \operatorname{Rad}_G(g_2)$.*

Proof. Suppose that there exists $r \in \mathrm{Rad}_G(g_1) \cap \mathrm{Rad}_G(g_2)$ such that $r \neq e$. This means that there are $a, b, m, n \in \mathbb{Z} \setminus \{0\}$ such that $g_1^a = r^m$ and $g_2^b = r^n$. Let $s \in \mathrm{Rad}_G(g_1)$ be arbitrary and let $c, k \in \mathbb{Z} \setminus \{0\}$ be such that $g_1^k = s^c$. It follows that

$$(s^c)^{an} = (g_1^k)^{an} = (g_1^a)^{kn} = (r^m)^{kn} = (r^n)^{km} = (g_2^b)^{kn}$$

and we see that $s \in \mathrm{Rad}_G(g_2)$. Consequently $\mathrm{Rad}_G(g_1) \subseteq \mathrm{Rad}_G(g_2)$. The opposite inclusion can be shown analogously, and therefore $\mathrm{Rad}_G(g_1) = \mathrm{Rad}_G(g_2)$. □

Proposition 9.2. *Let G be a torsion-free group. An element $g \in G$ is primitively stable with unique roots in G if and only if $\mathrm{Rad}_G(g)$ is an infinite cyclic subgroup of G.*

Proof. If g is primitively stable with unique roots, then exploiting Lemma 7.9 one can show that the element $\sqrt[G]{g}$ generates $\mathrm{Rad}_G(g)$.

Assume that $\mathrm{Rad}_G(g)$ is cyclic, and let $r \in \mathrm{Rad}_G(g)$ be a generator such that $g = r^n$ for some $n \in \mathbb{N} \setminus \{0\}$. We now show that $r = \sqrt[G]{g}$, that $n = \mathrm{plog}_G(g)$, and that g is primitively stable, in this order.

For this purpose, let $h_1, h_2 \in G$ and suppose that $h_1^m = g = h_2^m$ for some $m \in \mathbb{N}$. By definition $h_1, h_2 \in \mathrm{Rad}_G(g)$, therefore there are $a, b \in \mathbb{Z}$ such that $h_1 = r^a$ and $h_2 = r^b$. As $r^{am} = r^{bm}$ and r is torsion-free, we see that $am = bm$. Thus $a = b$, and in particular $h_1 = h_2$. This means that g has unique roots in G and $r = \sqrt[G]{g}$.

Assume now that $h \in G$ is such that $h^a = g$ for some $a \in \mathbb{N}$. By definition $h \in \mathrm{Rad}_G(g)$, and therefore $h = r^b$ for some $b \in \mathbb{Z}$. We see that

$$r^n = g = h^a = r^{ab}.$$

The element r has infinite order, so $ab = n$, hence $b > 0$ and $a \leq n$. It follows that n is maximal, that is $n = \mathrm{plog}_G(g)$.

Finally, we show that $\mathrm{plog}_G(g^m) = m \cdot \mathrm{plog}_G(g)$ for all $m \in \mathbb{N}$. Suppose that there is $h \in G$ such that $h^k = g^m$ for some $k \in \mathbb{N}$. Again, $h \in \mathrm{Rad}_G(g)$ so $h = r^a$ for some $a \in \mathbb{Z}$. As $g = r^n$, we see that $r^{nm} = r^{ak}$, hence $nm = ak$. It follows that $a > 0$ and that m divides kn. We see that $\mathrm{plog}_G(g^m) = nm = m\,\mathrm{plog}_G(g)$, and therefore g is primitively stable. □

In the light of this proposition, we say that g has *cyclic radical* in G whenever it is primitively stable and with unique root.

Note that Theorem C is an immediate consequence of Proposition 9.2

9.1. **Residually torsion-free nilpotent groups.** In this subsection we prove that residually finitely generated torsion-free nilpotent groups belong to $\mathfrak{U}_{\mathrm{ps}}$, that is they are primitively stable and with unique roots.

Lemma 9.3. *Let G be a torsion-free group let $g \in G \setminus \{e\}$ be arbitrary. Suppose that there is $N \trianglelefteq G$ such that $g \notin N$ and G/N is torsion-free, then the canonical projection $\pi \colon G \to G/N$ is injective on $\mathrm{Rad}_G(N)$, i.e. $\mathrm{Rad}_G(g) \cap N = \{e\}$.*

Proof. Let $r \in \mathrm{Rad}_G(g)$ be nontrivial, i.e. $r^a = g^b$ for some $a, b \in \mathbb{Z} \setminus \{0\}$. As $g \notin N$ and G/N is torsion-free, we see that $\pi(r)^a = \pi(g)^b \neq e$. It follows that $\pi(r) \neq e$. □

Proposition 9.4. *If G is a finitely generated torsion-free nilpotent group then $G \in \mathfrak{U}_{\mathrm{ps}}$.*

Proof. We will proceed by induction on the nilpotency class. If G is 1-step nilpotent, then G is a torsion-free abelian group. As G is finitely generated, we see that G is in fact free abelian. Clearly, free abelian groups belong to the class $\mathfrak{U}_{\mathrm{ps}}$.

Now suppose the statement holds for all finitely generated torsion-free groups of nilpotency class $n-1$, and suppose that G is n-step nilpotent. Let $\{e\} = Z_0 \leq \cdots \leq Z_n = G$ denote the upper central series of G and let $\pi_i \colon G \to G/Z_i$ denote the corresponding canonical projections. Recall that in the case of torsion-free nilpotent groups, the quotient G/Z_i is torsion-free for every $i \in \{0, \ldots, n-1\}$. Let $g \in G$ be non-trivial and let $r \in \mathrm{Rad}_G(g)$ be nontrivial as well, repeating the argument from Lemma 9.3 we see that $\pi_i(r)$ is trivial in G/Z_i if and only if $\pi_i(g)$ is. Pick $i \in \{0, \ldots, n-1\}$ such that $g \in Z_{i+1} \setminus Z_i$. We see that $\mathrm{Rad}_G(g) \setminus \{e\} \subseteq Z_{i+1} \setminus Z_i$.

Suppose that $i = 0$, i.e. $\mathrm{Rad}_G(g)$ is contained in Z_1, the center of G. As finitely generated nilpotent groups are slender, i.e. every subgroup is finitely generated, we see that the center Z_1 is finitely generated and therefore a free abelian group. We see that $\mathrm{Rad}_G(g) = \mathrm{Rad}_{Z_1}(g)$ and hence it must be cyclic.

If $i > 0$, then the group G/Z_i is a finitely generated group of nilpotency class $n - i$. By induction hypothesis we see that $\mathrm{Rad}_{G/Z_i}(\pi_i(g))$ is cyclic. Clearly, $\pi_i(\mathrm{Rad}_G(g)) \subseteq \mathrm{Rad}_{G/Z_i}(\pi_i(g))$. As π_i is injective on $\mathrm{Rad}_G(g)$ we see that g has a cyclic radical in G. □

Note that finite generation is essential in Proposition 9.4. Indeed, consider the group given by the presentation

$$G_p = \langle a_0, a_1, \cdots \parallel a_{i+1}^p = a_i \text{ for } i = 0, 1, \ldots \rangle \cong \mathbb{Z}[p^{-1}].$$

Obviously, G is a torsion-free abelian group, hence of nilpotency class one, and it is not finitely generated. It can be seen that $\mathrm{Rad}_G(a_0) = G$.

Combining Lemma 9.3 and Proposition 9.4 we immediately obtain the following result.

Corollary 9.5. *If G is a residually finitely generated torsion-free nilpotent group then $G \in \mathfrak{U}_{\mathrm{ps}}$.*

9.2. **Hyperbolic and relatively hyperbolic groups.** Recall that a group is called *elementary* if it contains a cyclic subgroup of finite index. The following was proved by Ol'shanskii [18, Lemma 1.16].

Lemma 9.6. *An infinite-order element g of a hyperbolic group G is contained in a unique maximal elementary subgroup, denoted by $\mathrm{E}_G(g)$.*

An element $x \in G$ belongs to $E_G(g)$ if and only if there exists $n \in \mathbb{Z} \setminus \{0\}$ such that $xg^n x^{-1} = g^{\pm n}$.

From this we can deduce that torsion-free hyperbolic groups belong to $\mathfrak{U}_{\mathrm{ps}}$.

Lemma 9.7. *If G is a torsion-free hyperbolic group then $G \in \mathfrak{U}_{\mathrm{ps}}$.*

Proof. Let $g \in G$ be non-trivial, and notice that $\mathrm{Rad}_G(g) \leq E_G(g)$. Indeed, let $r \in \mathrm{Rad}_G(g)$, so that there exist $a, b \in \mathbb{Z} \setminus \{0\}$ for which $r^a = g^b$. We have that

$$g^b = r^a = rg^br^{-1},$$

and by Lemma 9.6 we conclude that $r \in E_G(g)$. As G is torsion-free hyperbolic, we see that $\mathrm{E}_G(g)$ is cyclic. It can be easily seen that $\mathrm{Rad}_G(g)$ is in fact a subgroup of $\mathrm{E}_G(g)$ and, consequently, $\mathrm{Rad}_G(g)$ is an infinite cyclic subgroup of G. □

Let G be a group and let $\mathcal{H} = \{H_i \mid i \in I\}$ be a collection of subgroups of G, where I is a set. Then G is *hyperbolic relative to* $\mathcal{H}$ if the coned-off Cayley graph is hyperbolic and fine, in the sense of Bowditch [4]. We refer to [20] for more on relatively hyperbolic groups and equivalent definitions.

The following lemma is an easy corollary of [20, Theorem 1.4, Theorem 1.5].

Lemma 9.8. *Let G be torsion-free group and let $\mathcal{H} = \{H_i \mid i \in I\}$ be a collection of subgroups of G. If G is hyperbolic relative to $\mathcal{H}$ then the following are true:*

(i) *the set I is finite, i.e. $\mathcal{H} = \{H_1, \ldots, H_k\}$ for some $k \in \mathbb{N}$;*
(ii) *for any $g_1, g_2 \in G$ the intersection $H_i^{g_1} \cap H_j^{g_j}$ is trivial whenever $i \neq j$;*
(iii) *the intersection $H_i^g \cap H_i$ is trivial whenever $g \notin H_i$.*

Lemma 9.9. *Let G be torsion-free group and suppose that G is hyperbolic relative to a collection of subgroups $\mathcal{H} = \{H_1, \ldots, H_k\}$. If $g \in H_i$, then $\mathrm{Rad}_G(g) \leq H_i$, i.e. $\mathrm{Rad}_{H_i}(g) = \mathrm{Rad}_G(g)$.*

Proof. Let g be as above and let $r \in \mathrm{Rad}_G(g)$, i.e. $r^a = g^b$ for some $a, b \in \mathbb{Z} \setminus \{0\}$. Clearly, $rg^br^{-1} = rr^ar^{-1} = r^a = g^b$, so $\langle g^b \rangle \leq H_i \cap H_i^r$. Following Lemma 9.8, this is a contradiction unless $r \in H_i$. □

An element is said to be hyperbolic if its conjugacy class does not intersect any of the subgroups in $\mathcal{H}$. The following generalisation of Lemma 9.6 was proved in [19, Theorem 4.3].

Lemma 9.10. *Let G be hyperbolic relative to the family $\mathcal{H}$. Every hyperbolic element $g \in G$ of infinite order is contained in a unique maximal elementary subgroup $E_G(g)$, and moreover*

$$E_G(g) = \{x \in G \mid xg^nx^{-1} = g^{\pm n} \text{ for some } n \in \mathbb{Z} \setminus \{0\}\}.$$

From this we deduce:

Proposition 9.11. *Let G be a torsion-free group and suppose that G is hyperbolic relative to the collection of subgroups $\{H_1, \dots, H_k\}$. Then $G \in \mathfrak{U}_{\mathrm{ps}}$ if and only if $H_i \in \mathfrak{U}_{\mathrm{ps}}$ for $i = 1, \dots, k$.*

Proof. By Lemma 7.12 the class $\mathfrak{U}_{\mathrm{ps}}$ is closed under taking subgroups. Suppose therefore that $H_i \in \mathfrak{U}_{\mathrm{ps}}$ for $i = 1, \dots, k$, and let $g \in G$ be a non-trivial element. If g is hyperbolic then, following Lemma 9.10, $E_G(g)$ is cyclic and $\mathrm{Rad}_G(g) \leq E_G(g)$. Hence g has cyclic radical.

Suppose that $g \in H_i$. By Lemma 9.9 we see that $\mathrm{Rad}_G(g) = \mathrm{Rad}_{H_i}(g)$ which is cyclic by assumption. Finally, if $ygy^{-1} \in H_i$ for some $y \in G$, then $\mathrm{Rad}_G(g) = y \,\mathrm{Rad}_G(ygy^{-1})y^{-1}$, and therefore also in this case g has cyclic radical. □

Corollary 9.12. *If G is a toral relatively hyperbolic group then $G \in \mathfrak{U}_{\mathrm{ps}}$. In particular, if G is a limit group then $G \in \mathfrak{U}_{\mathrm{ps}}$.*

10. Separating cyclic subgroups of graph products in the pro-p topology

Proposition 10.1. *Free abelian groups are p-CSS.*

Proof. Let G be free abelian with free base $\{a_1, \dots, a_n\}$, and assume that $g = ({a_1}^{k_1}, \dots, {a_n}^{k_n}) \in G$ is p-isolated. We need to show that $\langle g \rangle$ is p-separable in G.

By Lemma 7.11 we have that $\mathrm{plog}_G(g) = \gcd(k_1, \dots, k_n) = p^e$ for some $e \in \mathbb{N}$. Let $f \in G$ be arbitrary and suppose that $f \notin \langle g \rangle$, that is $f = ({a_1}^{l_1}, \dots, {a_n}^{l_n})$ for some $l_1, \dots, l_n \in \mathbb{Z}$ and the system of equations

$$\mathcal{S} = \left\{ \begin{array}{c} k_1 z = l_1 \\ \vdots \\ k_n z = l_n \end{array} \right\}$$

has no integer solution for z.

Let $k := \prod_{i=1}^n k_i$ and $\overline{k_i} := k/k_i = \prod_{j \neq i} k_j$. For all $i = 1, \dots, n$, by multiplying the i-th equation of $\mathcal{S}$ by $\overline{k_i}$, we transform $\mathcal{S}$ into

$$\mathcal{S}' = \left\{ \begin{array}{c} kz = \overline{k}_1 l_1 \\ \vdots \\ kz = \overline{k}_n l_n \end{array} \right\}.$$

Clearly, $\mathcal{S}$ has a rational solution if and only if $\mathcal{S}'$ does, i.e. if and only if

$$\overline{k}_1 l_1 = \dots = \overline{k}_n l_n.$$

Thus, if $\mathcal{S}$ does not have a rational solution then there are $i, j \in \{1, \dots, n\}$ such that $\overline{k}_i l_i \neq \overline{k}_j l_j$. It can be easily seen that there is $r \in \mathbb{N}$ such that $\overline{k}_i l_i \not\equiv \overline{k}_j l_j \mod p^r$, i.e. the system $\mathcal{S}$ does not have a solution modulo p^r. It follows that $\pi(f) \notin \langle \pi(g) \rangle$, where $\pi \colon G \simeq \mathbb{Z}^n \to \mathbb{Z}^n_{p^r}$ is the natural projection modulo p^r.

Now suppose that $\mathcal{S}$ does have a rational solution. Note that if $k_i = 0$ then necessarily $l_i = 0$ and the equation $k_i = zl_i$ holds for any value of z. In this case, the given equation can be disregarded, hence we may assume that $k_i \neq 0$ for all $i \in \{1, \dots, n\}$. This means that the only possible $z \in \mathbb{Q}$ is given by

$$z = \frac{l_1}{k_1} = \cdots = \frac{l_n}{k_n}.$$

By the setting of the proof we have that $p^e = \gcd(k_1, \dots, k_n)$. Consider $l = \gcd(l_1, \dots, l_n)$, and notice that $l/p^e = l_i/k_i$ for every $i \in \{1, \dots, n\}$, or equivalently $z = l/p^e$. In particular, $k_i l = p^e l_i$ for $i \in \{1, \dots, n\}$.

If $p^e = 1$ then $l_i = k_i l$, i.e. $f = g^l$, which is a contradiction with $f \notin \langle g \rangle$. On the other hand, suppose that $e > 0$. As $k_i l = p^e l_i$ for $i \in \{1, \dots, n\}$, we see that $g^l = f^{p^e}$. It can be easily seen that there is $r \in \mathbb{N}$ big enough such that $\pi(f) \neq \pi(f)^{p^r}$, where $\pi \colon G \simeq \mathbb{Z}^n \to \bar{G} = \mathbb{Z}^n_{p^r}$ is the natural projection modulo p^r.

From the structure of the quotient $\bar{G}$, it is clear that both $\pi(f)$ and $\pi(g)$ are contained in a cyclic subgroup of order p^r, therefore either $\langle \pi(g) \rangle \subseteq \langle \pi(f) \rangle$ or $\langle \pi(f) \rangle \subseteq \langle \pi(g) \rangle$. We have that $\pi(f) \in \langle \pi(g) \rangle$ if and only if $\mathrm{ord}(\pi(f)) \leq \mathrm{ord}(\pi(g))$. Let $l = l' p^s$, where $p \nmid l'$. Clearly, $\mathrm{ord}(\pi(f)) = p^{r-s}$ and $\mathrm{ord}(\pi(g)) = p^{r-e}$. This means that $\pi(f) \in \langle \pi(g) \rangle$ if and only if $s > e$. However, if $s > e$ then we see that $f^l = g^{p^e}$ can be rewritten as

$$g^l = g^{l' p^e p^{s-e}} = \left(g^{l' p^{s-e}}\right)^{p^e} = f^{p^e},$$

hence $f = g^{l' p^{s-e}}$ which is a contradiction with $f \notin \langle g \rangle$.

We see that $\pi(f) \notin \langle \pi(g) \rangle$ and thus $\langle g \rangle$ is p-closed, that is p-separable, in G. $\square$

Lemma 10.2. *Let $G \in \mathfrak{U}_{\mathrm{ps}}$ be a p-CSS group, and suppose that $C \leq G$ is cyclic and p-isolated. Then* pro-$p(C)$ *is the restriction of* pro-$p(G)$.

Proof. Let $g \in G$ be a generator of C and set $r = \sqrt[G]{g}$. Following Lemma 7.10 we see that $\mathrm{plog}_G(g)$ is a power of p, i.e. $g = r^{p^e}$ for some $e \in \mathbb{N}$. As G is p-CSS, C is p-closed in G. Now let $N \in \mathcal{N}_p(C)$ be arbitrary. As C is infinite cyclic, the subgroup N is cyclic as well, let $f \in G$ be the generator. Clearly, $f = g^{p^k}$ for some $k \in \mathbb{N}$. We see that $f = g^{p^k} = (r^{p^e})^{p^k} = r^{p^{ek}}$. By primitive stability $\mathrm{plog}_G(f) = p^{ek}$, hence N is p-isolated in G by Lemma 7.10. Thus it is p-separable in G, being G a p-CSS group. The statement follows using Lemma 2.4. $\square$

Remark 10.3. Let G be a group and let H_1, H_2 be two subgroups such that $H_1 \leq H_2$. If H_1 is p-isolated in G then H_1 is p-isolated in H_2.

Lemma 10.4. *The class of primitively stable p-CSS groups with Unique Roots property is closed under forming direct products.*

Proof. Let $G_1, G_2 \in \mathfrak{U}_{\mathrm{ps}}$ be p-CSS groups, consider $G = G_1 \times G_2$, and note that $G \in \mathfrak{U}_{\mathrm{ps}}$ by Lemma 8.3. Let $g = (g_1, g_2) \in G$ be a p-isolated element. Set $r_i = \sqrt[G_i]{g_i}$ and denote $R_i = \langle r_i \rangle \leq G_i$, for $i = 1, 2$.

Note that $\mathrm{plog}_{G_1}(r_1) = 1 = \mathrm{plog}_{G_2}(r_2)$, hence R_1 is p-isolated in G_1 and R_2 is p-isolated in G_2 in view of Lemma 7.10. As both G_1 and G_2 are p-CSS, we see that R_1 is p-closed in G_1 and R_2 is p-closed in G_2.

Using Lemma 10.2 we see that pro-$p(R_1)$ is a restriction of pro-$p(G_1)$ and, similarly, pro-$p(R_2)$ is a restriction of pro-$p(G_2)$. It follows by Lemma 2.6 that pro-$p(R_1 \times R_2)$ is a restriction of pro-$p(G)$. As $R_1 \times R_2$ is a free abelian group on two generators, all its p-isolated cyclic subgroups are p-closed in $R_1 \times R_2$ by Proposition 10.1. Indeed, following Remark 10.3 we see that $\langle g \rangle$ is p-isolated in $R_1 \times R_2$. As pro-$p(R_1 \times R_2)$ is a restriction of pro-$p(G)$, we get that $\langle g \rangle$ is p-closed in G.

Therefore G is also p-CSS. $\square$

The following was proved in [3].

Theorem 10.5. *Let $G = G_1 *_R G_2$ be an amalgam over a common retract, let $g \in G$ be a p-isolated element in G, and suppose that G_1 and G_2 are residually p-finite. Then $\langle g \rangle$ is not p-separable in G if and only if g is conjugate to some $g_i \in G_i$, where $i \in \{1, 2\}$, and $\langle g_i \rangle$ is not p-separable in G_i.*

The following three statements provide a pro-p analogue of Lemma 5.2, Lemma 5.3 and Corollary 5.4, respectively. We omit the proofs, as they are more-or-less analogous.

Lemma 10.6. *Let $G = \Gamma\mathcal{G}$ be a graph product of residually p-finite groups and let $g \in G$ be arbitrary. Then the cyclic subgroup $\langle g \rangle \leq G$ is p-separable in G if and only if it is p-separable in G_S, where $S = \mathrm{supp}(g)$. Furthermore, $\langle g \rangle$ is p-separable in G if and only if $\langle g' \rangle$ is p-separable in G for some (and hence for all) $g' \in g^G$.*

Lemma 10.7. *Let $G = \Gamma\mathcal{G}$ be a graph product of residually p-finite groups and let $g \in G$ be a cyclically reduced element such that the full subgraph Γ_S contains a separating subset, where $S = \mathrm{supp}(g)$. Then the cyclic subgroup $\langle g \rangle \leq G$ is p-separable in G.*

Corollary 10.8. *Let Γ be a graph, $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be a family of residually p-finite groups and $G = \Gamma\mathcal{G}$ be the corresponding graph product. Suppose that $g \in G$ is an arbitrary element such that Γ_S contains a separating subset, where $S = \mathrm{esupp}(g)$. Then the cyclic subgroup $\langle g \rangle \leq G$ is p-separable in G.*

With all these, we can prove Theorem B.

Theorem B. For every prime number p, the class of p-CSS groups in $\mathfrak{U}_{\mathrm{ps}}$ is closed under forming graph products.

Proof. The proof of Theorem B is analogous to the proof of Theorem A, modulo the use of Theorem 8.6, Lemma 10.6, Lemma 10.7, and Corollary 10.8. □

11. p-isolated elements of graph products

The aim of this section is to give a full characterisation of p-isolated elements in graph products of groups in $\mathfrak{U}_{\mathrm{ps}}$.

11.1. **Irreducible factorisations in graph products of groups.** In this subsection we describe a canonical way to factorise elements in graph products into pairwise commuting factors that was introduced in [7] as the P-S decomposition.

Let $g \in G = \Gamma\mathcal{G}$ be an element in a graph product. We define $S(g) = \mathrm{supp}(g) \cap \mathrm{star}(\mathrm{supp}(g))$, where the star of a subset of vertices $A \subseteq V$ is defined as $\mathrm{star}(A) = \cap_{v\in A} \mathrm{star}(A)$. Similarly, we define $P(g) = \mathrm{supp}(g) \setminus S(g)$. The element g uniquely factorises as a reduced product $g = s(g)p(g)$, where $\mathrm{supp}(s(g)) = S(g)$ and $\mathrm{supp}(p(g)) = P(g)$. We call this factorisation the P-S decomposition of g.

Given a graph Γ, we consider the *complement graph* $\overline{\Gamma}$ of Γ, which is defined by $V\overline{\Gamma} = V\Gamma$ and $E\overline{\Gamma} = \binom{V\Gamma}{2} \setminus E\Gamma$. We say that a graph Γ is *irreducible* if $\overline{\Gamma}$ is connected, otherwise we say that Γ is reducible. Suppose that $\overline{\Gamma}$ can be split into a collection of disjoint connected components $C = \{\overline{\Gamma}_1, \overline{\Gamma}_2, \dots\}$, then the corresponding collection of full subgraphs $I = \{\Gamma_1, \Gamma_2, \dots\}$ is called the *irreducible decomposition* of Γ and the members of I are called *irreducible components* of Γ.

Suppose that Γ is a graph with at least two vertices and $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ is a family of non-trivial groups. If the graph is reducible, then the corresponding graph product $G = \Gamma\mathcal{G}$ splits as a direct product $G = \prod_{i\in I} G_i$ where i ranges over the collection of irreducible components of Γ. In fact, it was shown by Minasyan and Osin [16] that if the graph Γ is irreducible, then the corresponding graph product is an acylindrically hyperbolic group.

Let g be a non-trivial element of the graph product $\Gamma\mathcal{G}$, and consider the full subgraph $\Gamma_S \leq \Gamma$, where $S = \mathrm{supp}(g)$. Let $\{\Gamma_1, \dots, \Gamma_d\}$ be the irreducible decomposition of Γ_S, and let $G_1, \dots, G_d \leq G$ be the corresponding full subgroups. Then for every $1 \leq i \leq d$ there is a uniquely given $g_i \in G_i$ such that $g = g_1 \dots g_d$. We refer to this as the *irreducible factorisation* of g, and we call the individual elements g_i the *irreducible factors* of g. If $d = 1$, i.e. g has only one irreducible factor, then we say that g is *irreducible.* Note that $p(g)$ is the product of all the irreducible factors of lenght at least two and $s(g)$ is exactly the product of all irreducible factors of length one.

Lemma 11.1. *Let $G = \Gamma\mathcal{G}$ be a graph product and suppose that $g = g_1 \dots g_d$ is the irreducible factorisation of the element $g \in G$. The element g is cyclically reduced if and only if for every $i \in \{1, \dots, d\}$ exactly one of the following is true:*

(i) $|g_i| = 1$,
(ii) $\mathrm{FL}(g_i) \cap \mathrm{LL}(g_i) = \emptyset$.

Proof. Without loss of generality we can suppose that $\mathrm{supp}(g) = \Gamma$.

Suppose that there exists $j \in \{1, \dots, n\}$ such that $|g_j| > 1$ and $\mathrm{FL}(g_j) \cap \mathrm{LL}(g_j) \neq \emptyset$. Let $g_{j,\mathrm{first}}$ and $g_{j,\mathrm{last}}$ be respectively the first and last syllable of g_j: these two elements belong to the same vertex group G_v, for some vertex v. As $G = \prod_{i \in I} G_i$ is a direct product of the full subgroups induced by the irreducible components of Γ, by a sequence of syllable shufflings we see that

$$g \equiv (g_{j,\mathrm{first}}, \dots, g_{j,\mathrm{last}}). \tag{9}$$

As being cyclically reduced is independent of the word that represents the element, From Equation (9) we conclude that g is not cyclically reduced.

Suppose now that g is not cyclically reduced, and let

$$W \equiv (w_1, \dots, w_r), \qquad w_i \in G_{v_i} \tag{10}$$

be a reduced word representing g which witnesses g being not cyclically reduced. Therefore, the first syllable w_1 and the last syllable w_r belong to the same vertex group $G_{v_1} = G_{v_r}$. The syllables of the word W have the following property: w_i will commute with another syllable w_j, as soon as they belong to different full subgroups induced by the irreducible components of Γ. Therefore, by a sequence of syllable shufflings, from Equation (10) we obtain a new word $\overline{W}$ representing g:

$$\overline{W} \equiv (\bar{w}_{1,1}, \dots, \bar{w}_{1,r_1}, \dots, \bar{w}_{d,1}, \dots, \bar{w}_{d,r_d}),$$

where $\bar{w}_{i,r_j} \in G_{v_i}$ and v_i is a vertex belonging to the irreducible component Γ_i. As the irreducible factorisation of the element g is unique and W represents the element g, it must be that

$$g_i = \bar{w}_{i,1} \dots \bar{w}_{i,r_i}, \quad \forall\, i = 1, \dots, d.$$

Let Γ_j be the irreducible component where the vertex $v_1 = v_r$. Then $\bar{w}_{j,r_1} = w_1$ and $\bar{w}_{j,r_j} = w_r$, where w_1 and w_r are the first and the last syllable of the word in Equation (10). That is, we have that

$$g_j \equiv (w_1, \bar{w}_{j,2}, \dots, \bar{w}_{j,r_j-1}, w_r).$$

Therefore, from supposing that g is not cyclically reduced we conclude that condition *(ii)* is not satisfied. □

Lemma 11.2. *Let $G = \Gamma\mathcal{G}$ be a graph product and let $g \in G$ be irreducible and cyclically reduced. If $|g| > 1$ then $|g^n| = |n| \cdot |g|$ for every $n \in \mathbb{Z}$.*

Proof. The claim is clear if $n = 0$. Suppose that g is an irreducible element, and that n is positive. By Lemma 11.1 we know that $\mathrm{FL}(g) \cap \mathrm{LL}(g) = \emptyset$. As g is cyclically reduced, it is not possible to join together any pair of syllables belonging to consecutive appearances of g in g^n. Therefore $|g^n| = n \cdot |g|$ for all $n \geq 0$.

For n negative, we notice that $\mathrm{LL}(f^{-1}) = \mathrm{FL}(f)$ and $|f| = |f^{-1}|$ for every $f \in G$. □

Corollary 11.3. *Let $g \in G$ be irreducible and cyclically reduced. If $|g| > 1$ then* $\mathrm{ord}(g) = \infty$.

11.2. **p-isolated cyclic subgroups.** The aim of this subsection is to characterise the p-isolated elements of graph products of $\mathfrak{U}_{\mathrm{ps}}$-groups. We already proved in Theorem 8.6 that the class $\mathfrak{U}_{\mathrm{ps}}$ is closed under forming graph products. Following Lemma 7.10, an element is p-isolated if and only if its primitive logarithm is a power of p, so we only need to describe how to compute primitive logarithms. From now on we assume that G is a graph product of $\mathfrak{U}_{\mathrm{ps}}$ groups, and in particular that $G \in \mathfrak{U}_{\mathrm{ps}}$.

Lemma 11.4. *Let $g \in G$ be irreducible, cyclically reduced and let $S =$* $\mathrm{supp}(g)$*. Then* $\mathrm{plog}_G(g) = \mathrm{plog}_{G_S}(g)$*. Furthermore, if $|g| > 1$ then* $\mathrm{plog}_G(g)$ *divides $|g|$.*

Proof. The first part of the statement follows from Lemma 7.13.

Suppose that $|g| > 1$. If $\mathrm{plog}_G(g) = 1$ then it divides $|g|$. On the other hand, consider the case when $\mathrm{plog}_G(g) > 1$. As g is irreducible, cyclically reduced, and $g = r^{\mathrm{plog}_G(g)}$ where r is its root, it must necessarily be that $\mathrm{plog}_G(g)$ divides $|g|$. □

The previous lemma provides an informal algorithm to compute primitive logarithms and primitive roots of cyclically reduced irreducible elements (provided we can solve the word problem in every vertex group): given a reduced word $W_g = (g_1, \dots, g_n)$, where $g_i \in G_{v_i}$ for $i = 1, \dots, n$ and $v_i \in V\Gamma$, by shuffling we can construct the set of all reduced words representing g, denote it by $\mathcal{W}_g$, and then (in an increasing order) for every divisor d of n we can check whether there is $W \in \mathcal{W}_g$ with a prefix W_0 of length n/d such that $W_0^d =_G W_g$.

Lemma 11.5. *Let $g \in G$ be cyclically reduced and let $g = g_1 \dots g_s$ be its irreducible factorisation. Denote $S_i =$ $\mathrm{supp}(g_i)$ for $i = 1, \dots, s$ and $S = \mathrm{supp}(g)$. Then*

$$\mathrm{plog}_G(g) = \mathrm{plog}_{G_S}(g) = \gcd\left(\mathrm{plog}_{G_{S_1}}(g_1), \dots, \mathrm{plog}_{G_{S_s}}(g_s)\right).$$

Proof. The first equality is proven in Lemma 11.4. For the second, notice that $G_S = G_{S_1} \times \dots \times G_{S_s}$. By Lemma 7.12 each of the direct factors belongs to $\mathfrak{U}_{\mathrm{ps}}$, and therefore the rest of the statement follows from Lemma 8.1. □

Corollary 11.6. *Let $g \in G$ be cyclically reduced and let $g = g_1 \dots g_s$ be its irreducible factorisation. Denote $S_i =$ $\mathrm{supp}(g_i)$ for $i = 1, \dots, s$ and $S = \mathrm{supp}(g)$. Then g is p-isolated in G if and only if*

$$\gcd\left(\mathrm{plog}_{G_{S_1}}(g_1), \dots, \mathrm{plog}_{G_{S_s}}(g_s)\right) = p^e.$$

for some $e \in \mathbb{N}$.

Acknowledgements

The authors would like to thank Ashot Minasyan for many useful suggestions.

The initial stage of this work greatly benefited from the "Measured group theory" programme held in 2016 at the Erwin Schrödinger International Institute for Mathematics and Physics, Vienna. We thank the institution for its warm hospitality and for the excellent working environment. At that stage, the authors were supported by the ERC grant of Prof. Goulnara Arzhantseva, grant agreement no. 259527.

Federico Berlai is now supported in part by ERC grant PCG-336983, Basque Government Grant IT974-16, and Ministry of Economy, Industry and Competitiveness of the Spanish Government Grant MTM2017-86802-P.

Michal Ferov is now supported by the Australian Research Council Discovery Project grant DP160100486.

References

[1] V. G. Bardakov. On a question of D.I. Moldavanskiĭ's question about p-separable subgroups of a free group. *Siberian Mathematical Journal*, 45(3):416–419, 2004.

[2] F. Berlai and M. Ferov. Residual properties of graph products of groups. *Journal of Group Theory*, 19(2):217–231, 2016.

[3] P. A. Bobrovskii and E. V. Sokolov. The cyclic subgroup separability of certain generalized free products of two groups. *Algebra Colloq.*, 17(4):577–582, 2010.

[4] B. H. Bowditch. Relatively hyperbolic groups. *International Journal of Algebra and Computation*, 22(03):1250016, 2012.

[5] J. Burillo and A. Martino. Quasi-potency and cyclic subgroup separability. *Journal of Algebra*, 298(1):188 – 207, 2006.

[6] M. Ferov. *Separability properties of graph products of groups*. PhD thesis, University of Southampton, 2015.

[7] M. Ferov. On conjugacy separability of graph products of groups. *J. Algebra*, 447:135–182, 2016.

[8] E. R. Green. *Graph products of groups*. PhD thesis, University of Leeds, 1990.

[9] M. Hall, Jr. Subgroups of finite index in free groups. *Canad. J. Math*, 1(1):187–190, 1949.

[10] M. Hall, Jr. A topology for free groups and related groups. *Ann. of Math. (2)*, 52:127–139, 1950.

[11] R. C. Lyndon and P. E. Schupp. *Combinatorial group theory*. Springer-Verlag, Berlin, 1977. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89.

[12] W. Magnus, A. Karrass, and D. Solitar. *Combinatorial group theory: Presentations of groups in terms of generators and relations*. Courier Corporation, 2004.

[13] A. I. Mal'cev. On homomorphisms onto finite groups. *American Mathematical Society Translations, Series*, 2(119):67–79, 1983.

[14] K. A. Mihailova. The occurrence problem for free products of groups. *Mathematics of the USSR-Sbornik*, 4(2):181, 1968.

[15] A. Minasyan. Hereditary conjugacy separability of right-angled Artin groups and its applications. *Groups Geom. Dyn.*, 6(2):335–388, 2012.

[16] A. Minasyan and D. V. Osin. Acylindrical hyperbolicity of groups acting on trees. *Math. Ann.*, 362(3-4):1055–1105, 2015.

[17] A. Mostowski. On the decidability of some problems in special classes of groups. *Fund. Math.*, 59:123–135, 1966.

[18] A. Y. Ol'shanskii. On residualing homomorphisms and G-subgroups of hyperbolic groups. *International Journal of Algebra and Computation*, 3:365–410, 1993.

[19] D. V. Osin. Elementary subgroups of relatively hyperbolic groups and bounded generation. *International Journal of Algebra and Computation*, 16(01):99–118, 2006.

[20] D. V. Osin. *Relatively Hyperbolic Groups: Intrinsic Geometry, Algebraic Properties, and Algorithmic Problems*. Memoirs of the American Mathematical Society. American Mathematical Society, 2006.

[21] L. Ribes and P. Zalesskii. Profinite groups. In *Profinite Groups*, pages 19–77. Springer, 2000.

[22] E. Toinet. Conjugacy p-separability of right-angled Artin groups and applications. *Groups Geom. Dyn.*, 7(3):751–790, 2013.

(Federico Berlai) Department of Mathematics, UPV/EHU, Barrio Sarriena s/n, 48940, Leioa, Spain

E-mail address, Federico Berlai: `federico.berlai@ehu.eus`

(Michal Ferov) School of Mathematical and Physical Sciences, University of Technology Sydney, PO Box 123 Broadway, NSW 2007, Australia

E-mail address, Michal Ferov: `michal.ferov@gmail.com`