Forests of quasi-label-regular rooted trees

and their almost isomorphism classes

Ben Brawn

UoN

2019

Almost isomorphic

Definition

An *almost isomorphism* (S_1, S_2, φ) of locally-finite graphs $X_1(V_1, E_1)$ and $X_2(V_2, E_2)$ is a graph isomorphism $\varphi : X_1 \setminus S_1 \to X_2 \setminus S_2$ where S_i is a finite subset of edges and vertices of X_i. In this case we say X_1 and X_2 are *almost isomorphic*.

Almost isomorphic

Proposition

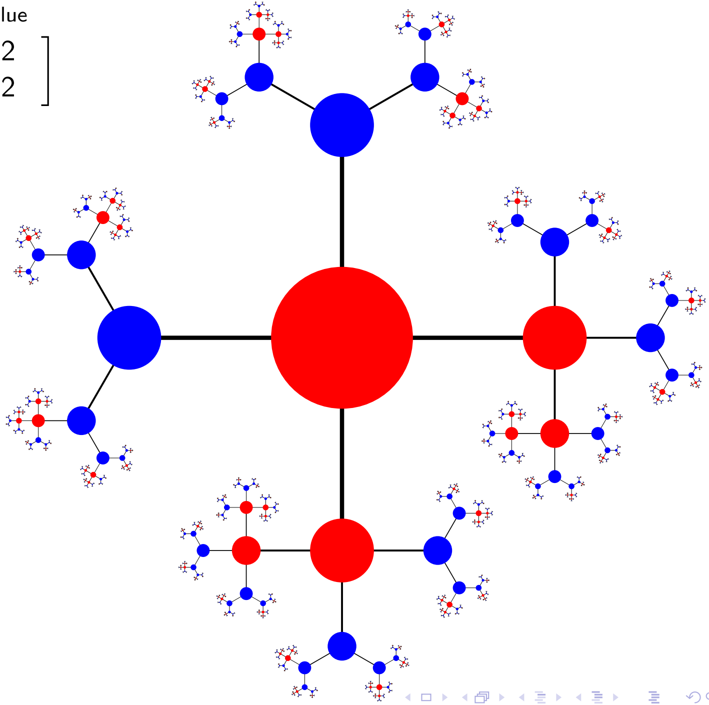
Two rooted infinite trees, $\mathcal{T}_1, \mathcal{T}_2$*, are almost isomorphic if and only if there exists a sequence of root removals for* $\mathcal{T}_1$ *and a sequence of root removals for* $\mathcal{T}_2$ *which result in two isomorphic forests* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *of rooted infinite trees.*

Label-regular trees

Definition

Let $T(V, E)$ be a locally finite tree with surjective labelling $l : V \to \{1, 2, \ldots, n\}$. Denote the set of neighbours of a vertex v by $B(v)$ where B is a set-valued function. Consider the multiset of labels of the neighbours of a vertex v, that is $l \circ B(v)$. If $l \circ B(v)$ only depends on $l(v)$, that is, $l(v_1) = l(v_2)$ implies $l \circ B(v_1) = l \circ B(v_2)$ for all vertices $v_1, v_2 \in V$, and T is infinite and has no leaves then we say T is a *label-regular tree*.

$$A = \begin{array}{c} \\ \text{red} \\ \text{blue} \end{array} \begin{array}{c} \begin{array}{cc} \text{red} & \text{blue} \end{array} \\ \left[\begin{array}{cc} 2 & 2 \\ 1 & 2 \end{array} \right] \end{array}$$



$$A = \begin{array}{c} \\ \text{red} \\ \text{blue} \end{array} \begin{array}{c} \begin{array}{cc} \text{red} & \text{blue} \end{array} \\ \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix} \end{array}$$

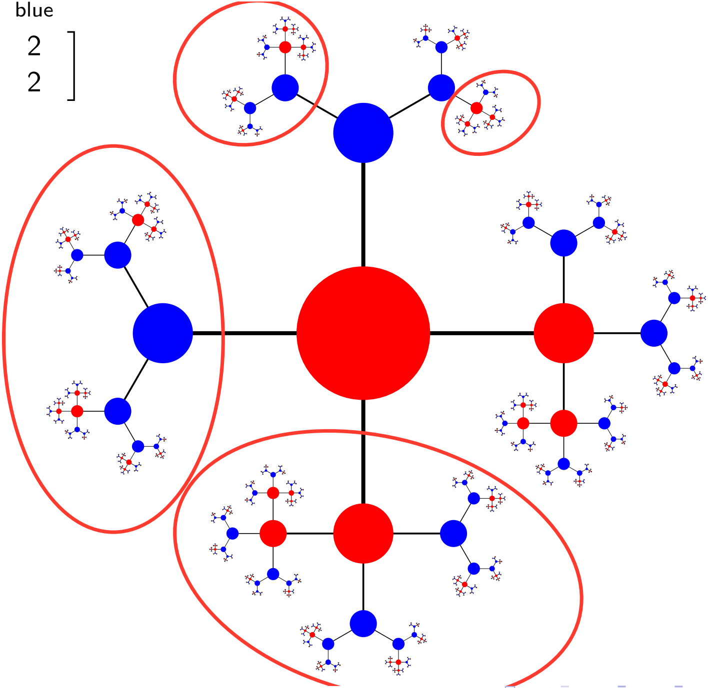
Rooted trees of types:

red \ red

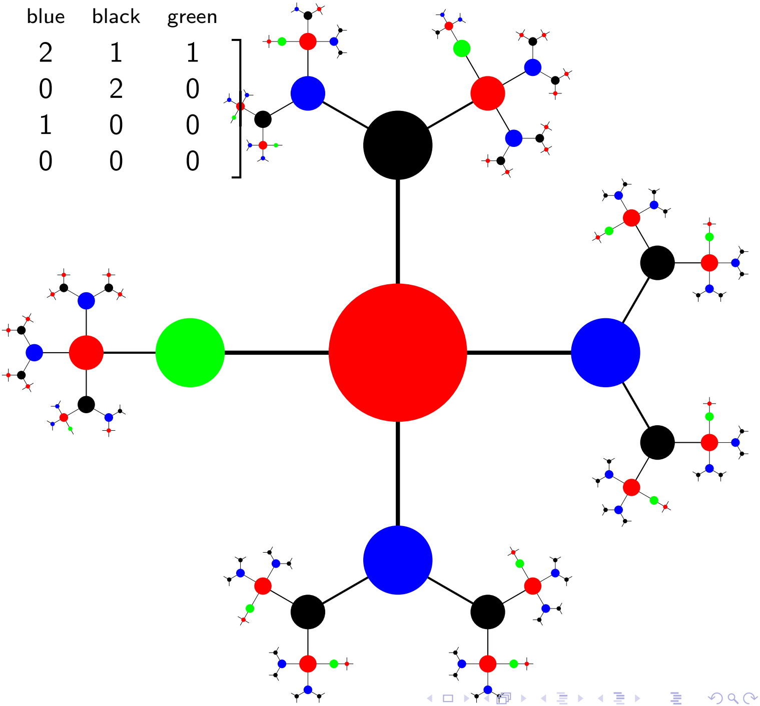
red \ blue

blue \ red

blue \ blue



$$A = \begin{array}{c} \\ \text{red} \\ \text{blue} \\ \text{black} \\ \text{green} \end{array} \begin{array}{c} \begin{array}{cccc} \text{red} & \text{blue} & \text{black} & \text{green} \end{array} \\ \begin{bmatrix} 0 & 2 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} \end{array}$$



Present

Definition

A rooted tree of type $i \setminus j$ is *present in* $\mathcal{T}_A$ if there exists an edge, e, connecting a vertex labelled i and a vertex labelled j, such that removing e results in a connected component that is a rooted tree of type $i \setminus j$. A rooted tree of type $i \setminus j$ is *present in* A if $a_{ij}, a_{ji} \neq 0$.

Forest vectors

Let A be an adjacency matrix with m non-zero entries. Let $\mathcal{F}_A$ be a forest of rooted trees with adjacency matrix A and let $\#(i \setminus j)$ be the number of connected components in $\mathcal{F}_A$ which are of type $i \setminus j$.

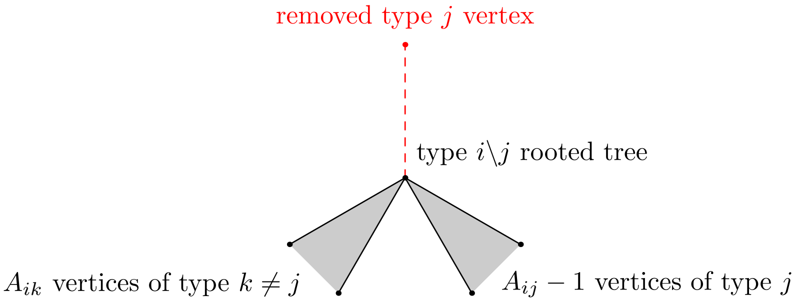
We describe $\mathcal{F}_A$ by the vector in $\mathbb{Z}^m_{\geq 0}$ with entries $\#(i \setminus j)$ ordered lexicographically on ij. We call this the *forest vector* of $\mathcal{F}_A$.

Forest vectors

For such an A, $\mathbb{Z}^m_{\geq 0}$ is the *space of forests*. Each point represents a forest of rooted trees. But some of these forests are almost isomorphic (i.e. we can remove roots until we are left with isomorphic forests).

Root removals

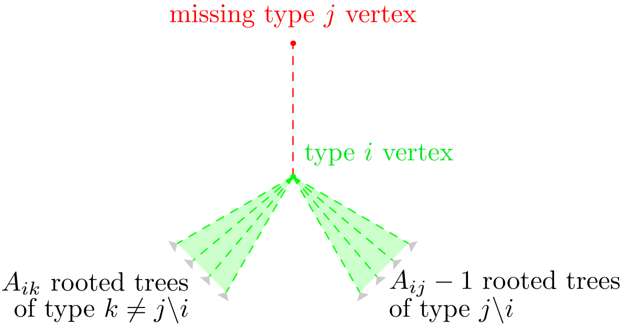
Consider the root of the rooted tree of type $i \backslash j$:



If we removed the root (labelled i) we would gain $A_{ij} - 1$ rooted trees of type $j \setminus i$, lose a rooted tree of type $i \setminus j$ and for each $k \neq j$ gain A_{ik} rooted trees of type $k \setminus i$.

Root removals

Consider the root of the rooted tree of type $i \backslash j$:



If we removed the root (labelled i) we would gain $A_{ij} - 1$ rooted trees of type $j \setminus i$, lose a rooted tree of type $i \setminus j$ and for each $k \neq j$ gain A_{ik} rooted trees of type $k \setminus i$.

Reduction vectors

For each type of rooted tree, say $i\backslash j$, we write down a *reduction vector*, $r_{i\backslash j}$, which when added to the forest vector emulates the effect of removing the root of a rooted tree of type $i\backslash j$.

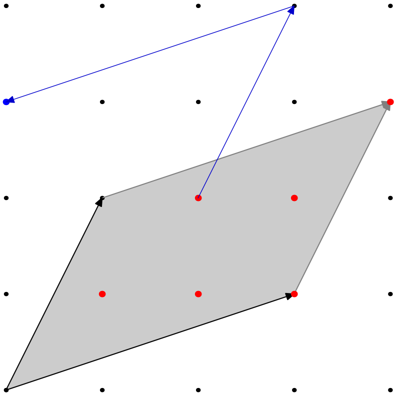
E.g. $A = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}$, $r_{1\backslash 1} = (0,0,2,0)$, $r_{1\backslash 2} = (2,-1,1,0)$, $r_{2\backslash 1} = (0,0,-1,2)$, $r_{2\backslash 2} = (0,1,0,0)$.

Proposition

Let $\mathcal{F}_{A,x}$ *and* $\mathcal{F}_{A,y}$ *be forests of rooted trees with forest vectors* x *and* y*.* $\mathcal{F}_{A,x}$ *and* $\mathcal{F}_{A,y}$ *are almost isomorphic if and only if there exists* $z_x, z_y \in \mathbb{Z}^m_{\geq 0}$ *such that* $x + Rz_x = y + Rz_y$*, where* R *is the matrix whose columns are the reduction vectors.*

$\det(R) \neq 0$

When the reduction vectors are linearly independent, we use the reduction vectors to form a half-open m-dimensional parallelotope. This is a fundamental domain in that every forest is almost isomorphic to a forest with forest vector in this region.



$\det(R) \neq 0$

That is, given any forest, with forest vector $x \in \mathbb{Z}^m$, we can:

1. write x in the basis $r_{i \setminus j}$,
2. reduce these coordinates mod 1 (i.e. $x \mapsto x - \lfloor x \rfloor$),
3. rewrite this in the standard basis, and

we will have a representative lattice point in the fundamental domain. That is, we will have a nice representative forest that is almost isomorphic to the original forest.

The number of integer points in this fundamental domain is $|\det(R)|$. However, often the reduction vectors are linearly dependent. In which case, there are an infinite number of almost isomorphically distinct forests, regardless, more can be said.

Well-mixed?

Definition

Let $P_{\mathcal{T}_A}$ denote the set of types of rooted trees present in a tree $\mathcal{T}_A$. Let P_A denote the set of types of rooted trees present in A. If for every $\Gamma_A \in P_A$ the set of types of rooted trees present in Γ_A, P_{Γ_A}, is equal to P_A, then we call T_A and A *well-mixed*.

Well-mixed?

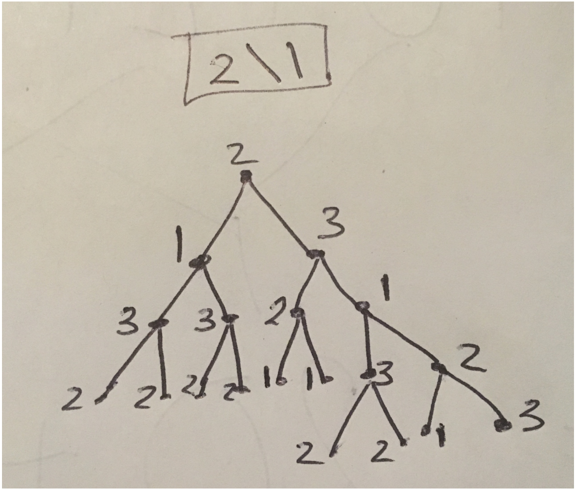
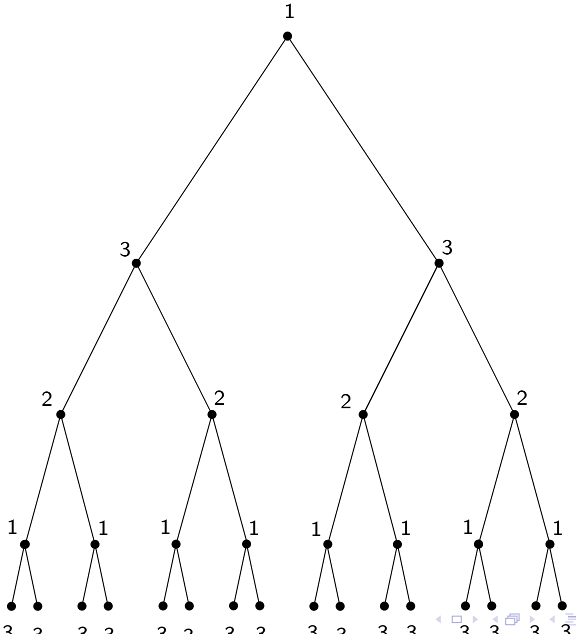


Figure: The rooted tree with $A = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}$ of type $2 \setminus 1$. $P_A = P_{2\setminus 1}$. Compare to $1 \setminus 2$.

Well-mixed?



M, $X(M)$

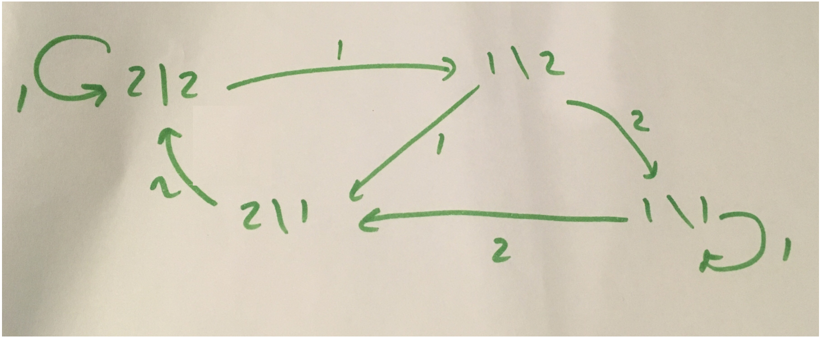
Definition

We define M to be a square matrix indexed by the m types of rooted trees that are present in an adjacency matrix A. The entry at position $i \setminus j, k \setminus l$ is the number of connected components that are rooted trees of type $k \setminus l$ after removing the root of a rooted tree of type $i \setminus j$.
We denote the weighted-directed graph associated with the matrix M, $X(M)$.

Note $M := R^T + I_m$.

M, $X(M)$

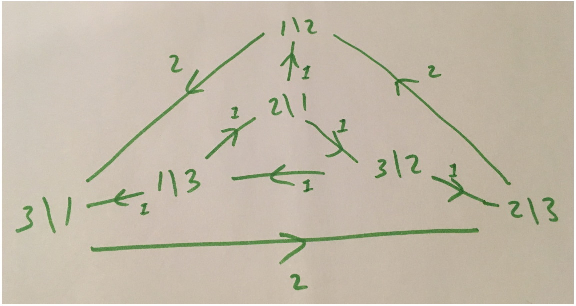
E.g. $A = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}$, $R = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 2 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}$, $M = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \end{pmatrix}$.



Here X(M) is strongly connected and A is well-mixed.

M, $X(M)$

$$\text{E.g. } A = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}, \quad M = \begin{array}{c} \\ 1\backslash 2 \\ 1\backslash 3 \\ 2\backslash 1 \\ 2\backslash 3 \\ 3\backslash 1 \\ 3\backslash 2 \end{array} \begin{array}{c} \begin{array}{cccccc} 1\backslash 2 & 1\backslash 3 & 2\backslash 1 & 2\backslash 3 & 3\backslash 1 & 3\backslash 2 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$



Here X(M) is not strongly connected and A is not well-mixed.

Well-mixed?

Lemma

A is well-mixed if and only if $X(M)$ is strongly connected (M is irreducible).

$X(M)$ in general

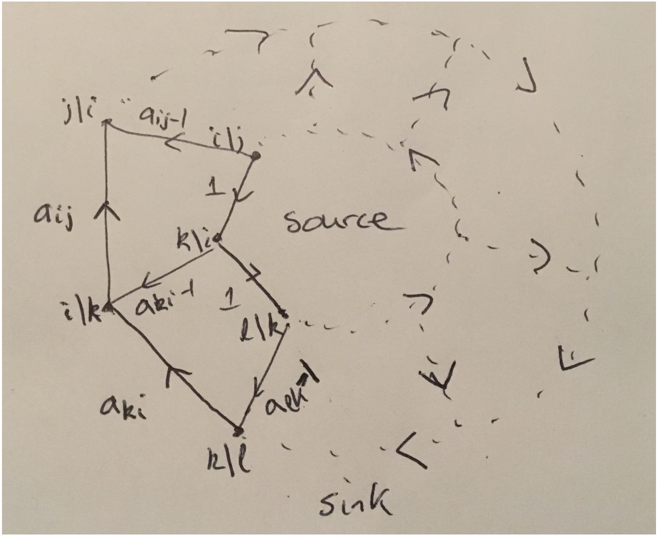


Figure: $X(M)$ when $\|A\|_\infty > 2$ and A is not well-mixed.

$X(M)$ in general

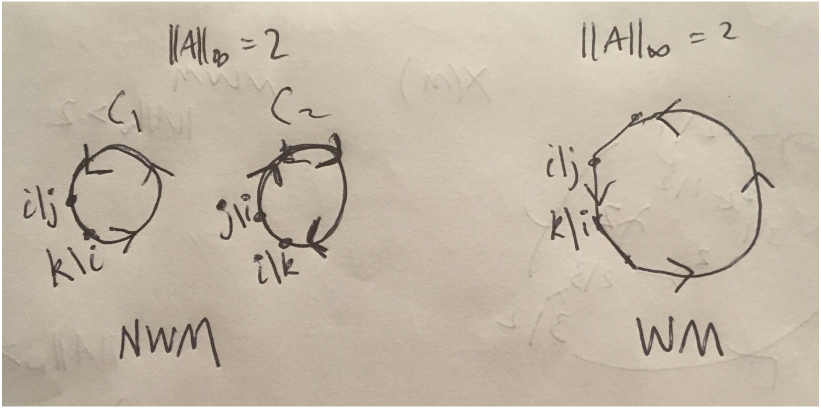


Figure: Left: $X(M)$ when $\|A\|_\infty = 2$ and A is not well-mixed. Right: $X(M)$ when $\|A\|_\infty = 2$ and A is well-mixed. Note all the edges have weight 1.

$X(M)$ in general

For the remaining case of A well-mixed and $\|A\|_\infty > 2$, you can find the 'period' of M, p, (the GCD of all the cycles in $X(M)$), then we can partition the vertices of $X(M)$ into p blocks B_i, such that a vertex in B_i must lead to a vertex in B_j where $j \equiv i + 1(\mathrm{mod} p)$.

Number of equivalence classes of forests (with a given A) under almost isomorphism

	Well-mixed A	Not well-mixed: At least 1 source tree	Not well-mixed: No source trees
$\|A\|_\infty = 2$ with k ends	~~Let $2 \le k \le \|\Omega_F\|$~~ for each k there is one ~~eq~~ class	for each k there are $k+1$ classes	for each k there is 1 class
$\|A\|_\infty > 2$ n # of ends	$\|\det(R)\|$ if $\neq 0$.	infinite number.	$\|\det(R')\| = \prod_{j \mid i \in \text{sink}} A_{ij} - 1$
	If $\det(R) = 0$ then an infinite number of classes	Fix the number of source trees; $\|\det(R')\| = \prod_{j \mid i \in \text{sink}} a_{ij} - 1$ (also gcd of $r \times r$ minors)	
	For a fixed $x^\perp$ there is the gcd of $r \times r$ minors where r is $\mathrm{rank}(R)$		

In another talk we will consider the automorphisms of rooted trees/forests and the almost automorphisms of forests.