An introduction to totally disconnected locally compact groups

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February 21, 2019

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Introduction

Preface

For G a locally compact group, the connected component which contains the identity, denoted by G°, is a closed normal subgroup. This observation produces a short exact sequence of topological groups

$$\{1\} \to G^\circ \to G \to G/G^\circ \to \{1\}$$

where G/G° is the group of left cosets of G° endowed with the quotient topology. The group G° is a connected locally compact group, and the group of left cosets G/G° is a totally disconnected locally compact (t.d.l.c.) group. The study of locally compact groups therefore in principle, although not always in practice, reduces to studying connected locally compact groups and t.d.l.c. groups.

The study of locally compact groups begins with the work [11] of S. Lie from the late 19th century. Since Lie's work, many deep and general results have been discovered for connected locally compact groups - i.e. the group G° appearing above. The quintessential example is, of course, the celebrated solution to Hilbert's fifth problem: *Connected locally compact groups are inverse limits of Lie groups.*

The t.d.l.c. groups, on the other hand, long resisted a general theory. There were several early, promising results due to D. van Dantzig and H. Abels, but these results largely failed to ignite an active program of research. The indifference of the mathematical community seems to have arose from an inability to find a coherent metamathematical perspective via which to view the many disparate examples, which were long known to include profinite groups, discrete groups, algebraic groups over non-archimedian local fields, and graph automorphism groups. The insight, first due to G. Willis [15] and later refined in the work of M. Burger and S. Mozes [3] and P.-E. Caprace and N. Monod [4], giving rise to a general theory is to consider t.d.l.c. groups as simultaneously *geometric groups and topological groups* and thus to investigate the connections between the geometric structure and the topological structure. This perspective gives the profinite groups and the discrete groups a special status as basic building blocks, since the profinite groups are trivial as geometric groups and the discrete groups are trivial as topological groups.

This book covers what I view as the fundamental results in the theory of t.d.l.c. groups. I aim to present in full and clear detail the basic theorems and techniques a graduate student or researcher will need to study t.d.l.c. groups.

Prerequisites

The reader should have the mathematical maturity of a second year graduate student. I assume a working knowledge of abstract algebra, point-set topology, and functional analysis. The ideal reader will have taken graduate courses in abstract algebra, point-set topology, and functional analysis.

Acknowledgments

These notes began as a long preliminary section of my thesis, and my thesis adviser Christian Rosendal deserves many thanks for inspiring this project. While a postdoc at Université catholique de Louvain, I gave a graduate course from these notes. David Hume, François Le Maître, Nicolas Radu, and Thierry Stulemeijer all took the course and gave detailed feedback on this text. François Le Maître deserves additional thanks for contributing to the chapter on Haar measures. During my time as a postdoc at Binghamton University, I again lectured from these notes. I happily thank Joshua Carey and Chance Rodriguez for their many suggestions.

Notes

Self-containment of text

This text is largely self-contained, but proving every fundamental background result would take us too far a field. Such results will always be stated as facts, as opposed to the usual theorem, proposition, or lemma. References for such facts will be given in the notes section of the relevant chapter. Theorems, propositions, and lemmas will always be proved, or the proofs will be left as exercises, when reasonable to do so.

Second countability

Most natural t.d.l.c. groups are second countable, but non-second countable t.d.l.c. groups unavoidably arise from time to time. One can usually deal with non-second countable groups by writing the group in question as a directed union of compactly generated open subgroups and reasoning about this directed system of subgroups. Compactly generated t.d.l.c. groups are second countable modulo a profinite normal subgroup, so as we consider profinite groups to be basic building blocks, these groups are effectively second countable. The theory of t.d.l.c. groups can thereby be reduced to studying second countable groups. In this text, we will thus assume our groups are second countable whenever convenient.

In the setting of locally compact groups, second countability admits a useful characterization.

Definition. A topological space is **Polish** if it is separable and admits a complete metric which induces the topology.

Fact. *The following are equivalent for a locally compact group G:*

(1) G is Polish.

(2) G is second countable.

(3) G is metrizable and K_σ - i.e. has a countable exhaustion by compact sets.

Locally compact second countable groups are thus exactly the locally compact members of the class of Polish groups, often studied in descriptive set theory, permutation group theory, and model theory. We may in particular use the term “Polish” in place of “second countable” in the setting of locally compact groups, and we often do so for two reasons. First and most practically, we will from time to time require classical results for Polish groups, and this will let us avoid rehashing the above fact. Second, the study of t.d.l.c. groups seems naturally situated within the study of Polish groups, and we wish to draw attention to this fact.

Basic definitions

A **topological group** is a group endowed with a topology such that the group operations are continuous. All topological groups and spaces are taken to be Hausdorff.

For G a topological group acting on a topological space X, we say that the action is **continuous** if the action map $G \times X \to X$ is continuous, where $G \times X$ is given the product topology.

Notations

We use "t.d.", "l.c.", and "s.c." for "totally disconnected", "locally compact", and "second countable", respectively. We write $\forall^\infty$ to stand for the phrase "for all but finitely many." We use $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{Z}$ for the complex numbers, rationals, reals, and the integers, respectively. We use the notation $\mathbb{R}_{\geq 0}$ to denote the non-negative real numbers.

For H a closed subgroup of a topological group G, G/H denotes the space of left cosets, and $H\backslash G$ denotes the space of right cosets. We primarily consider left coset spaces. All quotients of topological groups are given the quotient topology.

The center of G is denoted by $Z(G)$. For any subset $K \subseteq G$, $C_G(K)$ is the collection of elements of G that centralize every element of K. We denote the collection of elements of G that normalize K by $N_G(K)$. The topological closure of K in G is denoted by $\overline{K}$.

For $A, B \subseteq G$, we put

$$\begin{gathered} A^B := \{bab^{-1} \mid a \in A \text{ and } b \in B\}, \\ [A,B] := \langle aba^{-1}b^{-1} \mid a \in A \text{ and } b \in B\rangle \text{ and} \\ A^n := \{a_1 \dots a_n \mid a_i \in A\}. \end{gathered}$$

For $k \geq 1$, $A^{\times k}$ denotes the k-th Cartesian power. For $a, b \in G$, $[a,b] := aba^{-1}b^{-1}$.

We denote a group G acting on a set X by $G \curvearrowright X$. Groups are always taken to act on the left. For a subset $F \subseteq X$, we denote the pointwise stabilizer of F in G by $G_{(F)}$. The setwise stabilizer is denoted by $G_{\{F\}}$.

Chapter 1

Topological Structure

Our study of t.d.l.c. groups begins with an exploration of the topology. This chapter aims to give an understanding of how these groups look from a topological perspective as well as to develop basic techniques to manipulate the topology.

1.1 van Dantzig's theorem

In a topological group, the topology is determined by a neighborhood basis at the identity. Having a collection $\{U_\alpha\}_{\alpha\in I}$ of arbitrarily small neighborhoods of the identity, we obtain a collection of arbitrarily small neighborhoods of any other group element g by forming $\{gU_\alpha\}_{\alpha\in I}$. We thus obtain a basis

$$\mathcal{B} := \{gU_\alpha \mid g \in G \text{ and } \alpha \in I\}$$

for the topology on G. We, somewhat abusively, call the collection $\{U_\alpha\}_{\alpha\in I}$ a **basis of identity neighborhoods** for G.

The topology of a totally disconnected locally compact (t.d.l.c.) group admits a well-behaved basis of identity neighborhoods; there is no need for the Polish assumption here. Isolating this basis requires a couple of classical results from point-set topology.

A topological space is **totally disconnected** if every connected subset has at most one element. A space is **zero dimensional** if it admits a basis of clopen sets; a **clopen** set is both closed and open. Zero dimensional spaces are totally disconnected, but in general the converse does not hold. For locally compact spaces, however, the converse does hold, by classical results. Recall that we take all topological spaces to be Hausdorff.

Lemma 1.1. *Let X be a compact space.*

1. *If C and D are non-empty closed subsets such that $C \cap D = \emptyset$, then there are disjoint open sets U and V such that $C \subseteq U$ and $D \subseteq V$. That is, X is a **normal** topological space.*

2. *If $x \in X$ and A is the intersection of all clopen subsets of X containing x, then A is connected.*

Proof. For (1), let us first fix $c \in C$. Since X is Hausdorff, for each $d \in D$, there are disjoint open sets O_d and P_d such that $c \in O_d$ and $d \in P_d$. The set D is compact, so there is a finite collection $d_1, \ldots, d_n$ of elements of D such that $D \subseteq \bigcup_{i=1}^{n} P_{d_i}$. We now see that $U_c := \bigcap_{i=1}^{n} O_{d_i}$ and $V_c := \bigcup_{i=1}^{n} P_{d_i}$ are disjoint open sets such that $c \in U_c$ and $D \subseteq V_c$.

For each $c \in C$, the set U_c is an open set that contains c, and V_c is an open set that is disjoint from U_c with $D \subseteq V_c$. As C is compact, there is a finite collection $c_1, \ldots, c_m$ of elements of C such that $C \subseteq U := \bigcup_{i=1}^{m} U_{c_i}$. On the other hand, $D \subseteq V_{c_i}$ for each $1 \leq i \leq m$, so $D \subseteq V := \bigcap_{i=1}^{m} V_{c_i}$. The sets U and V satisfy the claim.

For (2), suppose that $A = C \cup D$ with C and D open in A and $C \cap D = \emptyset$; note that both C and D are closed in X. Applying part (1), we may find disjoint open sets U and V of X such that $C \subseteq U$ and $D \subseteq V$. Let $\{C_\alpha \mid \alpha \in I\}$ list the set of clopen sets of X that contain x. The intersection

$$\bigcap_{\alpha \in I} C_\alpha \cap (X \setminus (U \cup V))$$

is empty, so there is some finite collection $\alpha_1, \ldots, \alpha_k$ in I such that $H := \bigcap_{i=1}^{n} C_{\alpha_i} \subseteq U \cup V$. We may thus write $H = H \cap U \cup H \cap V$, and since H is clopen, both $H \cap U$ and $H \cap V$ are clopen. The element x must be a member of one of $H \cap U$ or $H \cap V$; without loss of generality, we assume $x \in H \cap U$. The set A is the intersection of all clopen sets that contain x, so $A \subseteq H \cap U$. The set A is then disjoint from V which contains D, so D is empty. We conclude that A is connected. □

Lemma 1.2. *A totally disconnected locally compact space X is zero dimensional.*

Proof. Say that X is a totally disconnected locally compact space. Let $O \subseteq X$ be a compact neighborhood of $x \in X$ and say that $x \in U \subseteq O$ with U

open in X. The set O is a totally disconnected compact space under the subspace topology.

Letting $\{C_i\}_{i\in I}$ list clopen sets of O containing x, Lemma 1.1 ensures that $\bigcap_{i\in I} C_i = \{x\}$. The intersection $\bigcap_{i\in I} C_i \cap (O \setminus U)$ is empty, so there is $i_1, \dots, i_k$ such that $\bigcap_{j=1}^k C_{i_j} \subseteq U$. The set $\bigcap_{j=1}^k C_{i_j}$ is closed in O, so it is closed in X. On the other hand, $\bigcap_{j=1}^k C_{i_j}$ is open in the subspace topology on O, so there is $V \subseteq G$ open such that $V \cap O = V \cap U = \bigcap_{j=1}^k C_{i_j}$. We conclude that $\bigcap_{j=1}^k C_{i_j}$ is clopen in X. Hence, X is zero dimensional. □

That t.d.l.c. spaces are zero dimensional gives a canonical basis of identity neighborhoods for a t.d.l.c. group.

Theorem 1.3 (van Dantzig)**.** *A t.d.l.c. group admits a basis at* 1 *of compact open subgroups.*

Proof. Let V be a compact neighborhood of 1 in G. By Lemma 1.2, G admits a basis of clopen sets at 1, so we may find $U \subseteq V$ a compact open neighborhood of 1. We may further take U to be symmetric since the inversion map is continuous.

For each $x \in U$, there is an open set W_x containing 1 with $xW_x \subseteq U$ and an open symmetric set L_x containing 1 with $L_x^2 \subseteq W_x$. The compactness of U ensures that $U \subseteq x_1L_{x_1} \cup \dots \cup x_kL_{x_k}$ for some $x_1, \dots, x_k$. Putting $L := \bigcap_{i=1}^k L_{x_i}$, we have

$$UL \subseteq \bigcup_{i=1}^k x_iL_{x_i}L \subseteq \bigcup_{i=1}^k x_iL_{x_i}^2 \subseteq \bigcup_{i=1}^k x_iW_{x_i} \subseteq U$$

We conclude that $UL \subseteq U$. Furthermore, induction on n shows that $L^n \subseteq U$ for all $n \geq 0$: if $L^n \subseteq U$, then $L^{n+1} = L^nL \subseteq UL \subseteq U$.

The union $W := \bigcup_{n\geq 0} L^n$ is thus contained in U. Since L is symmetric, W is an open subgroup of the compact open set U. As the complement of any open subgroup is open, W is indeed clopen, and therefore, W is a compact open subgroup of G contained in V. The theorem now follows. □

We may now deduce several corollaries.

Notation 1.4. For a t.d.l.c. group G, we denote the collection of compact open subgroups by $\mathcal{U}(G)$.

We are primarily interested in t.d.l.c. Polish groups, and for such groups G, van Dantzig's theorem ensures that $\mathcal{U}(G)$ is small.

Corollary 1.5. *If G is a t.d.l.c. Polish group, then $\mathcal{U}(G)$ is countable.*

Proof. Since G is second countable, we may fix a countable dense subset D of G. Applying van Dantzig's theorem, we may additionally fix a decreasing sequence $(U_i)_{i\in\mathbb{N}}$ of compact open subgroups giving a basis of identity neighborhoods.

For $V \in \mathcal{U}(G)$, there is i such that $U_i \leq V$. The subgroup V is compact, so U_i is of finite index in V. We may then find coset representatives $v_1, \dots, v_m$ such that $V = \bigcup_{j=1}^m v_jU_i$. For each v_j, there is $d_j \in D$ for which $d_j \in v_jU_i$, since the set D is dense in G. Therefore, $V = \bigcup_{j=1}^m d_jU_i$. We conclude that $\mathcal{U}(G)$ is contained in the collection of subgroups which are generated by $U_i \cup F$ for some $i \in \mathbb{N}$ and finite $F \subseteq D$. Hence, $\mathcal{U}(G)$ is countable. □

A much more striking corollary of van Dantzig's theorem is that there are very few homeomorphism types of the underlying topological space. This requires the first fact of the main text. Recall that we will not prove facts; however, the notes section directs the reader to a proof.

A topological space is called **perfect** if it has no isolated points.

Fact 1.6 (Brouwer)**.** *Any two non-empty compact Polish spaces which are perfect and zero dimensional are homeomorphic to each other.*

The Cantor space, denoted by $\mathcal{C}$, is thus the unique Polish space that is compact, perfect, and totally disconnected.

Brouwer's theorem along with van Dantzig's theorem allow us to identify exactly the homeomorphism types of t.d.l.c. Polish groups.

Corollary 1.7. *For G a t.d.l.c. Polish group, one of the following hold:*

1. *G is homeomorphic to an at most countable discrete topological space.*
2. *G is homeomorphic to $\mathcal{C}$.*
3. *G is homeomorphic to $\mathcal{C} \times \mathbb{N}$ with the product topology.*

Proof. If the topology on G is discrete, then G is at most countable, since it is Polish, so (1) holds. Let us suppose that G is non-discrete. If the topology on G is compact, then G is perfect, compact, and totally disconnected; see

Exercise 1.2. In this case, (2) holds. Let us then suppose that G is neither discrete nor compact. Via Theorem 1.3, we obtain a compact open subgroup $U \leq G$. The group G is second countable, so we can fix coset representatives $(g_i)_{i\in\mathbb{N}}$ such that $G = \bigsqcup_{i\in\mathbb{N}} g_iU$. For each $i \in \mathbb{N}$, the coset g_iU is a perfect, compact, and totally disconnected topological space, so Brouwer's theorem tells us that g_iU and $\mathcal{C}$ are homeomorphic. Fixing a homeomorphism $\phi_i : g_iU \to \mathcal{C}$ for each $i \in \mathbb{N}$, one verifies the map $\phi : G \to \mathcal{C} \times \mathbb{N}$ by $\phi(x) := (\phi_i(x), i)$ when $x \in g_iU$ is a homeomorphism. Hence, (3) holds. $\square$

Remark 1.8. Contrary to the setting of connected locally compact groups, Corollary 1.7 shows that there is no hope of using primarily the topology to investigate the structure of a t.d.l.c. Polish group. One must consider the algebraic structure and, as will be introduced later, the geometric structure in an essential way.

Finally, profinite groups, i.e. inverse limits of finite groups, form a fundamental class of topological groups, and from van Dantzig's theorem, one can easily deduce that any compact totally disconnected group is profinite. The converse, that every profinite group is totally disconnected and compact, follows from the definition of a profinite group. The notes section of this Chapter gives references for these facts.

Corollary 1.9. *A t.d.l.c. group admits a basis at* 1 *of open profinite subgroups.*

1.2 Isomorphism theorems

The usual isomorphism theorems for groups hold in the setting of l.c. groups under some natural modifications, which are necessary to account for the topological structure. The most important of these modifications will be the assumption that the groups are Polish. The reason for this modification will be to allow us to apply the classical Baire category theorem, which we will take as one of our facts.

Let X be a Polish space and $N \subseteq X$. We say N is **nowhere dense** if $\overline{N}$ has empty interior. We say $M \subseteq X$ is **meagre** if M is a countable union of nowhere dense sets.

Fact 1.10 (Baire Category Theorem)**.** *If X is a Polish space and $U \subseteq X$ is a non-empty open set, then U is non-meagre.*

Recall that an **epimorphism** from a group G to a group H is a surjective homomorphism.

Theorem 1.11 (First isomorphism theorem). *Suppose that G and H are t.d.l.c. Polish groups and $\phi : G \to H$ is a continuous epimorphism. Then ϕ is an open map. Furthermore, the induced map $\tilde{\phi} : G/\ker(\phi) \to H$ given by $g\ker(\phi) \mapsto \phi(g)$ is an isomorphism of topological groups.*

Proof. Suppose that $B \subseteq G$ is open and fix $x \in B$. We may find $U \in \mathcal{U}(G)$ such that $xU \subseteq B$. If $\phi(U)$ is open, then $\phi(xU) = \phi(x)\phi(U) \subseteq \phi(B)$ is open. The map ϕ is thus open if $\phi(U)$ is open for every $U \in \mathcal{U}(G)$.

Fix $U \in \mathcal{U}(G)$. As G is second countable, we may find $(g_i)_{i\in\mathbb{N}}$ a countable set of left coset representatives for U in G. Hence,

$$H = \bigcup_{i\in\mathbb{N}} \phi(g_iU) = \bigcup_{i\in\mathbb{N}} \phi(g_i)\phi(U).$$

The subgroup U is compact, so $\phi(g_iU)$ is closed. The Baire category theorem then implies that $\phi(g_i)\phi(U)$ is non-meagre for some i. Multiplication by $\phi(g_i)$ is a homeomorphism of G, so $\phi(U)$ is non-meagre. The subgroup $\phi(U)$ thus has a non-empty interior, and it follows from Exercise 1.5 that $\phi(U)$ is open in H.

For the second claim, it suffices to show $\tilde{\phi}$ is continuous since $\tilde{\phi}$ is bijective and our previous discussion ensures it is an open map. Taking $O \subseteq H$ open, $\phi^{-1}(O)$ is open, since ϕ is continuous. Letting $\pi : G \to G/\ker(\phi)$ be the usual projection, the map π is a an open map, via Exercise 1.4, so

$$\pi(\phi^{-1}(O)) = \tilde{\phi}^{-1}(O)$$

is open in $G/\ker(\phi)$. Hence, $\tilde{\phi}$ is continuous. □

As a corollary to the first isomorphism theorem, we do not need to check that inverses are continuous to verify that a given isomorphism is also a homeomorphism.

Corollary 1.12. *Suppose that G is a t.d.l.c. Polish group. If $\psi : G \to G$ is a continuous group isomorphism, then ψ is an automorphism of G as a topological group. That is, ψ^{-1} is also continuous.*

We now move on to the second isomorphism theorem. This theorem requires a mild technical assumption in addition to the requirement that our groups are Polish. The new assumption is that a certain subgroup is closed. This ensures that the subgroup in question is locally compact when given the subspace topology and thereby allows us to apply the first isomorphism theorem.

Theorem 1.13 (Second isomorphism theorem)**.** *Suppose that G is a t.d.l.c. Polish group, $A \leq G$ is a closed subgroup, and $H \trianglelefteq G$ is a closed normal subgroup. If AH is closed, then $AH/A \simeq A/A \cap H$ as topological groups.*

Proof. Give AH and A the subspace topology and let $\iota : A \to AH$ be the obvious inclusion. The map ι is continuous. Letting $\pi : AH \to AH/H$ be the projection $x \mapsto xH$, the map π is a continuous epimorphism between t.d.l.c. Polish groups, so the composition $\pi \circ \iota : A \to AH/H$ is a continuous epimorphism. The first isomorphism theorem now implies $A/A\cap H \simeq AH/H$ as topological groups. □

We finally verify the third isomorphism theorem.

Theorem 1.14 (Third isomorphism theorem)**.** *Suppose that G and H are t.d.l.c. Polish groups with $\phi : G \to H$ a continuous epimorphism. If $N \trianglelefteq H$ is a closed normal subgroup, then $G/\phi^{-1}(N)$, H/N, and*

$$(G/\ker(\phi))\,/\,\big(\phi^{-1}(N)/\ker(\phi)\big)$$

are all isomorphic as topological groups.

Proof. Let $\pi : H \to H/N$ be the usual projection; note that π is open and continuous. Applying the first isomorphism theorem to $\pi \circ \phi : G \to H/N$, we deduce that $G/\phi^{-1}(N) \simeq H/N$ as topological groups.

Applying the first isomorphism theorem to $\phi : G \to H$, the induced map $\tilde{\phi} : G/\ker(\phi) \to H$ is an isomorphism of topological groups. The composition $\pi \circ \tilde{\phi}$ is thus a continuous epimorphism with $\ker(\pi \circ \tilde{\phi}) = \phi^{-1}(N)/\ker(\phi)$. We conclude that

$$(G/\ker(\phi))\,/\,\big(\phi^{-1}(N)/\ker(\phi)\big) \simeq H/N$$

as topological groups. □

Let us close this subsection with a useful characterization of continuous homomorphisms. Recall that a function $f : X \to Y$ between topological spaces is **continuous at** $x \in X$ if for every open neighborhood V of $f(x)$

there is an open neighborhood U of x such that $f(U) \subseteq V$. It is easy to verify that a function is continuous if and only if it is continuous at every point.

The homogeneity of topological groups allows for a substantial weakening of the continuity at every point characterization of continuity. Indeed, one only need check continuity at 1.

Proposition 1.15. *Suppose that G and H are topological groups with $\phi : G \to H$ a continuous homomorphism. Then ϕ is continuous if and only if ϕ is continuous at* 1*.*

Proof. The forward implication is immediate. Conversely, suppose ϕ is continuous at 1, fix $g \in G$, and let V be an open neighborhood of $\phi(g)$ in H. The translate $\phi(g^{-1})V$ is then an open neighborhood of 1, so we may find U an open set containing 1 such that $\phi(U) \leq \phi(g^{-1})V$. The set gU is an open neighborhood of g, and moreover,

$$\phi(gU) \leq \phi(g)\phi(g^{-1})V \leq V.$$

We conclude that ϕ is continuous at every $g \in G$, so ϕ is continuous. $\square$

Proposition 1.15 can be improved further for t.d.l.c. groups. See Exercise 1.19.

1.3 Graph automorphism groups

Automorphism groups of locally finite connected graphs give a large and natural family of examples of t.d.l.c. groups. We discuss these examples now because they give proper intuition for the topological structure of t.d.l.c. groups, and the reader is encouraged to keep these in mind as we progress through the text. In fact, we will see in Chapter 3 that these examples are integral to the theory of t.d.l.c. groups: *All* compactly generated t.d.l.c. groups lie in this class of examples.

Definition 1.16. A **graph** Γ is a pair $(V\Gamma, E\Gamma)$ where $V\Gamma$ is a set and $E\Gamma$ is a collection of unordered distinct pairs of elements from $V\Gamma$. We call $V\Gamma$ the set of **vertices** of Γ and $E\Gamma$ the set of **edges** of Γ.

Remark 1.17. One may alternatively consider, as logicians often do, $E\Gamma$ as a relation on $V\Gamma$. We discourage this perspective, because we shall later, in Chapter 4, need to modify our definition of a graph to allow for multiple edges and loops. This modification will be a natural extension of the definition given here.

We will need a variety of terminology for graphs. For a vertex v, we define $E(v)$ to be the collection of edges e such that $v \in e$. A graph is **locally finite** if $|E(v)| < \infty$ for every $v \in V\Gamma$.

A **path** p is a sequence of vertices $(p(0), \cdots, p(n))$ such that $\{p(i), p(i+1)\} \in E\Gamma$ for each $i < n$. The length of p, denoted by $l(p)$, is n. The length counts the number of edges used in the path. The **origin** of p, denoted by $o(p)$, is the first term of the path, i.e. $p(0)$. The **terminus** of p, denoted by $t(\gamma)$, is the last term of the path, i.e. $p(l(p))$. A least length path between two vertices is called a **geodesic**. A graph is **connected** if there is a path between any two vertices. Connected graphs are metric spaces under the graph metric: The **graph metric** on a connected graph Γ is

$$d_\Gamma(v, u) := \min \{l(p) \mid p \text{ is a path connecting } v \text{ to } u\}.$$

For $v \in V\Gamma$ and $k \geq 0$, the k**-ball** around v is defined to be

$$B_k(v) := \{w \in V\Gamma \mid d_\Gamma(v, w) \leq k\}$$

and the k**-sphere** is

$$S_k(v) := \{w \in V\Gamma \mid d_\Gamma(v, w) = k\}.$$

When we wish to emphasize the graph in which we are taking $B_k(v)$ and $S_k(v)$, we write $B_k^\Gamma(v)$ and $S_k^\Gamma(v)$.

For graphs Γ and Δ, a **graph homomorphism** is a function $\psi : V\Gamma \to V\Delta$ such that if $\{v, w\} \in E\Gamma$, then $\{f(v), f(w)\} \in E\Delta$. A **graph isomorphism** is a bijection $\psi : V\Gamma \to V\Delta$ such that $\{g(v), g(w)\} \in E\Delta$ if and only if $\{v, w\} \in E\Gamma$. An **automorphism** of a graph Γ is an isomorphism $\psi : \Gamma \to \Gamma$. The collection of automorphisms forms a group, and it is denoted by $\mathrm{Aut}(\Gamma)$. When Γ is a connected graph, the automorphism group is the same as the isometry group of Γ, when Γ is regarded as a metric space under the graph metric.

We now identify a topology under which graph automorphism groups are topological groups. We stress that this topology is not locally compact in general. Let Γ be a graph. For finite tuples $\overline{a} := (a_1, \dots, a_n)$ and $\overline{b} := (b_1, \dots, b_n)$ of vertices of Γ, define

$$\Sigma_{\overline{a},\overline{b}} := \{g \in \mathrm{Aut}(\Gamma) \mid g(a_i) = b_i \text{ for } 1 \leq i \leq n\}.$$

The collection $\mathcal{B}$ of sets $\Sigma_{\overline{a},\overline{b}}$ as $\overline{a}$ and $\overline{b}$ run over finite sequences of vertices forms a basis for a topology on $\mathrm{Aut}(\Gamma)$. The topology generated by $\mathcal{B}$ is called the **pointwise convergence topology**; exercise 1.25 motivates this terminology. The pointwise convergence topology is sometimes called the **permutation topology**, but we will not use this term.

Proposition 1.18. *Let Γ be a graph. Equipped with the pointwise convergence topology,* $\mathrm{Aut}(\Gamma)$ *is a topological group.*

Proof. Set $G := \mathrm{Aut}(\Gamma)$. We need to show that composition and inversion in G are continuous under the pointwise convergence topology.

For inversion, take a basic open set $\Sigma_{\overline{a},\overline{b}}$ and consider the inversion map $i : G \to G$, which is defined by $i(g) := g^{-1}$. The preimage of $\Sigma_{\overline{a},\overline{b}}$ under the inversion map is $\Sigma_{\overline{b},\overline{a}}$. As $\Sigma_{\overline{b},\overline{a}}$ is an open set, inversion is continuous.

Let $m : G \times G \to G$ be the multiplication map and fix a basic open set $\Sigma_{\overline{a},\overline{b}}$ with $\overline{a} = (a_1, \dots, a_n)$ and $\overline{b} = (b_1, \dots, b_n)$. Fix $(h, g) \in m^{-1}(\Sigma_{\overline{a},\overline{b}})$. There is a tuple $\overline{c} = (c_1, \dots, c_n)$ such that $g(a_i) = c_i$, and since $hg \in \Sigma_{\overline{a},\overline{b}}$, it must be the case that $h(c_i) = b_i$. The open set $\Sigma_{\overline{c},\overline{b}} \times \Sigma_{\overline{a},\overline{c}}$ is then an open set containing (h, g), and it is contained in $m^{-1}(\Sigma_{\overline{a},\overline{b}})$. The preimage $m^{-1}(\Sigma_{\overline{a},\overline{b}})$ is thus open, so m is continuous. $\square$

Remark 1.19. We shall always assume the automorphism group of a graph is equipped with pointwise convergence topology.

From the definition of the pointwise convergence topology, we immediately deduce the following.

Proposition 1.20. *Let Γ be a graph. If Γ is countable, then the pointwise convergence topology on* $\mathrm{Aut}(\Gamma)$ *is second countable.*

Let us now explore the properties of the topological group $\mathrm{Aut}(\Gamma)$ further. Set $G := \mathrm{Aut}(\Gamma)$ and for $F \subseteq V\Gamma$ finite, define $G_{(F)}$ to be the pointwise stabilizer of the set F in G. The set $G_{(F)}$ is a basic open set, and

$$\mathcal{F} := \{G_{(F)} \mid F \subseteq V\Gamma \text{ with } |F| < \infty\}$$

is a basis at the identity. The sets $G_{(F)}$ are subgroups, so $\mathcal{F}$ in fact is a basis of clopen subgroups; see Exercise 1.5. Since a basis for the topology on $\mathrm{Aut}(\Gamma)$ is given by cosets of the elements of $\mathcal{F}$, we have proved the following proposition.

Proposition 1.21. *Let Γ be a graph. The pointwise convergence topology on $\mathrm{Aut}(\Gamma)$ is zero dimensional. In particular, the pointwise convergence topology is totally disconnected.*

The last property we here explore identifies a natural condition which ensures that $\mathrm{Aut}(\Gamma)$ is a locally compact group. The proof requires a useful notion of a Cauchy sequence in a topological group. While topological groups do not have a metric in general, there is nonetheless a notion of "close together" as we can consider the "difference" of two group elements. Indeed, even in the presence of a metric, this notion of a Cauchy sequence is often more useful for topological arguments.

Definition 1.22. Let G be a topological group and $\mathcal{B}$ a basis of identity neighborhoods. A sequence $(g_i)_{i\in\mathbb{N}}$ of elements of G is a **Cauchy sequence** if for every $B\in\mathcal{B}$ there is $N\in\mathbb{N}$ such that $g_i^{-1}g_j\in B$ and $g_jg_i^{-1}\in B$ for all $i,j\geq N$.

The reader should work Exercise 1.8, which verifies that the definition of a Cauchy sequence does not depend on the choice of $\mathcal{B}$.

Lemma 1.23. *Let Γ be a graph. If $(g_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{Aut}(\Gamma)$, then there is $g\in\mathrm{Aut}(\Gamma)$ such that $(g_i)_{i\in\mathbb{N}}$ converges to g.*

Proof. For each $v\in V\Gamma$, fix $N_v\geq 1$ such that $g_i^{-1}g_j(v)=v$ and $g_ig_j^{-1}(v)=v$ for all $i,j\geq N_v$. We may find such an N_v since $(g_i)_{i\in\mathbb{N}}$ is a Cauchy sequence. Define a function $g: V\Gamma\to V\Gamma$ by $g(v):=g_{N_v}(v)$.

We first argue that g is a permutation of $V\Gamma$. Suppose that $g(v)=g(w)$. Thus, $g_{N_v}(v)=g_{N_w}(w)$. Taking $M:=\max\{N_v,N_w\}$, we see that $g_M(v)=g_{N_v}(v)$ and $g_M(w)=g_{N_w}(w)$. Thus, $g_M(v)=g_M(w)$, and since g_M is a bijection, $v=w$. The function g is therefore injective.

To see that g is surjective, take $w\in V\Gamma$. Observe that $g_ig_j^{-1}(w)=w$ for all $i,j\geq N_w$ and set $v:=g_j^{-1}(w)$ for some fixed $j\geq N_w$. Taking M greater than both N_v and N_w, we have

$$g_M(v)=g_{N_v}(v)=g(v)$$

and

$$g_M(v) = g_M g_j^{-1}(w) = w$$

We conclude that $g(v) = w$, and thus, g is also surjective.

Let us now see that g also respects the graph structure. Fix $\{v, w\}$ a distinct pair of vertices. Taking M greater than both N_v and N_w. We see that $g(v) = g_{N_v}(v) = g_M(v)$ and $g(w) = g_{N_w}(w) = g_M(w)$. Since g_M preserves the graph structure, we conclude that $\{v, w\} \in E\Gamma$ if and only if $\{g(v), g(w)\} \in E\Gamma$. Hence, $g \in \mathrm{Aut}(\Gamma)$.

We leave that $g_i \to g$ as Exercise 1.26. □

Theorem 1.24. *Let Γ be a graph. If Γ is locally finite and connected, then $\mathrm{Aut}(\Gamma)$ is a t.d.l.c. Polish group.*

Proof. Set $G := \mathrm{Aut}(\Gamma)$, fix a vertex $v \in V\Gamma$, and take the vertex stabilizer $G_{(v)}$. For each $k \geq 0$, set $S_k := S_k(v)$, where $S_k(v)$ is the k-sphere around v. Since Γ is locally finite, it follows by induction on k that S_k is finite for every $k \geq 0$.

The group $G_{(v)}$ acts on each S_k as a permutation, so we obtain a family of homomorphisms $\phi_k : G \to \mathrm{Sym}(S_k)$. Define $\Phi : G_{(v)} \to \prod_{k\geq 1} \mathrm{Sym}(S_k)$ by $\Phi(g) := (\phi_k(g))_{k\geq 1}$. The map Φ is a homomorphism, since each ϕ_k is a homomorphism. As Γ is connected, $V\Gamma = \bigcup_{k\geq 0} S_k$, so Φ is also injective.

Equipping the group $\prod_{k\geq 1} \mathrm{Sym}(S_k)$ with the product topology produces a compact topological group; we give the finite groups $\mathrm{Sym}(S_k)$ the discrete topology. In view of Proposition 1.15, it suffices to check that Φ is continuous at 1 to verify that Φ is continuous. Setting $L := \prod_{k\geq 1} \mathrm{Sym}(S_k)$, a basis at 1 for L is given by the subgroups

$$\Delta_n := \{(r_i)_{i\geq 1} \in L \mid r_i = 1 \text{ for all } i \leq n\}.$$

Fixing $n \geq 1$, the pointwise stabilizer $G_{(\bigcup_{i=1}^n S_i)}$ is an open subgroup of $G_{(v)}$, and moreover, $\Phi(U) \leq \Delta_n$. The map Φ is thus continuous at 1, so Φ is continuous.

We next argue that Φ is a closed map. Take $A \subseteq G_{(v)}$ a closed set and suppose that $(\Phi(a_i))_{i\in\mathbb{N}}$ with $a_i \in A$ is a convergent sequence in L. The sequence $(\Phi(a_i))_{i\in\mathbb{N}}$ is a Cauchy sequence in L, so for every Δ_n, there is N such that $\Phi(a_i^{-1}a_j) \in \Delta_n$ and $\Phi(a_j a_i^{-1}) \in \Delta_n$ for any $i, j \geq N$. The elements $a_j a_i^{-1}$ and $a_i^{-1}a_j$ therefore fix $B_n(v)$ pointwise for all $i, j \geq N$. As the collection of pointwise stabilizers $\{G_{(B_n(v))} \mid n \geq 1\}$ form a basis at 1, we deduce that $(a_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in $G_{(v)}$. Lemma 1.23 now supplies

$a \in G$ such that $a_i \to a$. As A is closed, we indeed have that $a \in A$. Since Φ is continuous, we infer that $\Phi(a) = \lim_i \Phi(a_i)$, so $\Phi(A)$ is closed. The map Φ is thus a closed map.

The homomorphism Φ is a topological group isomorphism. Hence, $G_{(v)}$ is isomorphic to a closed subgroup of the compact group L and is compact. We conclude that G is locally compact. Propositions 1.21 and 1.20 ensure that G is in fact a t.d.l.c. Polish group. $\square$

Theorem 1.24 establishes a natural, general condition under which $\mathrm{Aut}(\Gamma)$ is a t.d.l.c. group. One naturally asks if one can find a general condition to ensure that $\mathrm{Aut}(\Gamma)$ is non-discrete. Unfortunately, there are no such general conditions, and worse, it is a notoriously difficult problem to determine if the automorphism group of a given locally finite connected graph is non-discrete. There are somewhat special, but nonetheless compelling geometric conditions on graphs which make it much easier to find non-discrete graph automorphism groups. A source of such examples are given by locally finite trees: A **tree** is a connected graph such that there are no cycles. A **cycle** is a path $p_1, \dots, p_n$ with $n > 1$ such that $p_i = p_j$ if and only if $\{i, j\} = \{1, n\}$. It is easy to produce many examples of locally finite trees with non-discrete automorphism groups; see Exercise 1.28.

Notes

Corollary 1.7 is particularly striking if one is accustomed to connected Lie groups. Information about to topology of a connected Lie group often gives deep insight into the structure of the group. For instance, two compact connected simple Lie groups are isomorphic as Lie groups if and only if they have the same homotopy type (see [2, Theorem 9.3]).

For an excellent discussion and proof of the Baire category theorem as well as a proof of Brouwer's theorem, the reader is directed to [9]. For the statements of the isomorphism theorems in full generality, we refer the reader to [8].

Profinite groups - i.e. inverse limits of finite groups - are exactly the compact t.d.l.c. groups, and the theory of these groups has a beautiful interplay with the theory of finite groups. For an introduction to profinite groups, we recommend the wonderful text by J. Wilson [16]. While we do not discuss profinte groups in this text, a good understanding of profinite groups is necessary to work in t.d.l.c. group theory, as they play an essential role in the theory.

1.4 Exercises

Topological groups

Exercise 1.1. Let G be a group with normal subgroups L and K. Show if $L \cap K = \{1\}$, then L and K centralize each other.

Exercise 1.2. Let G be a topological group. Show G is perfect as a topological space if and only if G is non-discrete.

Exercise 1.3. Suppose that G is a topological group and V is an open neighborhood of g. Show there is an open set W containing 1 such that $WgW \subseteq V$.

Exercise 1.4. Let G be a topological group and N a closed normal subgroup. Show the projection $\pi : G \to G/N$ is an open map.

Exercise 1.5. Let G be a topological group and $H \leq G$ a subgroup with non-empty interior. Show H is open and closed in G.

Exercise 1.6. Let G be a topological group and $H \trianglelefteq G$ be a normal subgroup. Show $\overline{H}$ is normal subgroup of G.

Exercise 1.7. Let G and H be topological groups and $\phi : G \to H$ be a homomorphism. Show that ϕ is continuous if and only if ϕ is continuous at some $g \in G$.

Exercise 1.8. Show the definition of a Cauchy sequence does not depend on the choice of a basis of identity neighborhoods.

Exercise 1.9. Let G be a topological group and suppose $(g_i)_{i\in\mathbb{N}}$ is a convergent sequence. Show $(g_i)_{i\in\mathbb{N}}$ is a Cauchy sequence.

Exercise 1.10. Let G be a locally compact group. Show that every Cauchy sequence in G converges to a limit in G.

Locally compact groups

Exercise 1.11. Let G be a locally compact group. Show the connected component of the identity is a closed normal subgroup of G.

Exercise 1.12. Let G be a non-compact and non-discrete t.d.l.c. Polish group. Fix compact open subgroup $U \leq G$, coset representatives $(g_i)_{i\in\mathbb{N}}$ such that $G = \bigsqcup_{i\in\mathbb{N}} g_iU$, and a homeomorphism $\phi_i : g_iU \to \mathcal{C}$ for each $i \in \mathbb{N}$, where $\mathcal{C}$ is the Cantor set. Verify the map $\phi : G \to \mathcal{C}\times\mathbb{N}$ by $\phi(x) := (\phi_i(x), i)$ when $x \in g_iU$ is a homeomorphism.

Exercise 1.13. Let G be a compact t.d.l.c. group. Show G admits a basis at 1 of compact open *normal* subgroups and each of these subgroups is of finite index in G.

Exercise 1.14. Suppose that G is a t.d.l.c. group with $K \leq G$ a compact subgroup. Show there is a compact open subgroup U of G containing K.

Exercise 1.15. Suppose that G is a t.d.l.c. group and $K \trianglelefteq G$ is such that K and G/K are compact. Show G is compact. This shows the class of profinite groups - i.e. compact t.d.l.c. groups - is closed under group extension.

Exercise 1.16. Suppose that G is a t.d.l.c. Polish group, $K \subseteq G$ is compact, and $C \subseteq G$ is closed. Show KC is a closed subset of G.

Exercise 1.17. Let G be a t.d.l.c. Polish group, $U \in \mathcal{U}(G)$, and $K \subseteq G$. Show $\overline{K\cap U} = \overline{K}\cap U$.

Exercise 1.18. Suppose G is a t.d.l.c. Polish group, $H \leq G$, and $U \in \mathcal{U}(G)$. Show if $H\cap U$ is closed in G, then H is closed in G.

Exercise 1.19. Let G and H be t.d.l.c. groups and $\psi : G \to H$ be a homomorphism. Show that if the preimage under ψ of every open subgroup of H is open in G, then ψ is continuous.

Exercise 1.20. Find t.d.l.c. groups G and H and a continuous isomorphism $\psi : G \to H$ of which is not an isomorphism of topological groups. HINT: G and H cannot be Polish groups.

Exercise 1.21. Suppose $(G_i)_{i\in\mathbb{N}}$ is a countable $\subseteq$-increasing sequence of t.d.l.c. Polish groups such that G_i is an open subgroup of G_{i+1} for each i. Define a collection τ of subsets of $\bigcup_{i\in\mathbb{N}} G_i$ by $A \in \tau$ if and only if $A\cap G_i$ is open in G_i for all $i \in \mathbb{N}$. Show τ is a topology on $\bigcup_{i\in I} G_i$ and verify $\bigcup_{i\in\mathbb{N}} G_i$ is a t.d.l.c. Polish group under this topology. We call τ the **inductive limit topology**.

Exercise 1.22. Suppose that G is a t.d.l.c. group with a continuous action on a countable set X, where X is equipped with the discrete topology. Show $A \subseteq G$ is relatively compact - i.e., $\overline{A}$ is compact - if and only if for all $x \in X$ the set $A.x := \{a.x \mid a \in A\}$ is finite.

Graphs

Exercise 1.23. Let Γ be a graph and for any finite tuples $\overline{a} := (a_1, \dots, a_n)$ and $\overline{b} := (b_1, \dots, b_n)$ of vertices, define

$$\Sigma_{\overline{a},\overline{b}} := \{g \in \mathrm{Aut}(\Gamma) \mid g(a_i) = b_i \text{ for } 1 \leq i \leq n\}.$$

Show the collection $\mathcal{B}$ of the sets $\Sigma_{\overline{a},\overline{b}}$ as $\overline{a}$ and $\overline{b}$ range over finite tuples of vertices is a basis for a topology on $\mathrm{Aut}(\Gamma)$.

Exercise 1.24. Let Γ be a locally finite connected graph and fix $v \in V\Gamma$. For $k \geq 1$, show $B_k(v) := \{w \in V\Gamma \mid d_\Gamma(v, w) \leq k\}$ is finite.

Exercise 1.25. Let Γ be a graph and and $(g_i)_{i\in\mathbb{N}}$ be a sequence of elements from $\mathrm{Aut}(\Gamma)$. Show $(g_i)_{i\in\mathbb{N}}$ converges to some $g \in \mathrm{Aut}(\Gamma)$ if and only if for every finite set $F \subset V\Gamma$ there is N such that $g_i(x) = g(x)$ for all $x \in F$ and all $i \geq N$.

Exercise 1.26. Show that $g_i \to g$ for $(g_i)_{i\in\mathbb{N}}$ and g as defined in Lemma 1.23.

Exercise 1.27. Use the proof of Theorem 1.24 to give an alternative proof of van Dantzig's theorem for automorphism groups of locally finite connected graphs.

Exercise 1.28. For $n \geq 3$, let T_n be the tree such that $\deg(v) = n$ for every $v \in VT$; we call T_n the n**-regular tree**. Show $\mathrm{Aut}(T_n)$ is non-discrete.

Chapter 2

Haar Measure

The primary goal of this chapter is to establish the existence and uniqueness of a canonical measure on t.d.l.c. Polish groups, called the Haar measure. The Haar measure is a powerful tool, which allows the techniques from functional analysis to be applied to the study of locally compact groups. We here restrict to t.d.l.c. Polish groups, but the Haar measure exists on any locally compact group. We make this restriction as the proofs become easier.

2.1 Functional analysis

We will need the basics of functional analysis, but we will restrict our discussion to Polish spaces.

Fix a Polish space X. The collection of continuous functions $f : X \to \mathbb{C}$ is denoted by $C(X)$. The set $C(X)$ is a vector space under pointwise addition and scalar multiplication. The **support** of $f \in C(X)$ is

$$\mathrm{supp}(f) := \{x \in X \mid f(x) \neq 0\}.$$

A function $f \in C(X)$ is **compactly supported** if $\overline{\mathrm{supp}(f)}$ is compact. We denote the set of compactly supported functions in $C(X)$ by $C_c(X)$. A function $f \in C(X)$ is said to be **positive** if $f(X) \subseteq \mathbb{R}_{\geq 0}$. The collection of all positive functions in $C_c(X)$ is denoted by $C_c^+(X)$.

We can equip $C_c(X)$ with the following norm:

$$\|f\| := \max\{|f(x)| \mid x \in X\}.$$

This norm is called the **uniform norm** on $C_c(X)$. It induces a metric d on $C_c(X)$ defined by $d(f,g) := \|f-g\|$. The metric topology on $C_c(X)$ turns $C_c(X)$ into a topological vector space. That is to say, the vector space operations are continuous in this topology; see Exercise 2.5. We call this topology the **uniform topology**, and we shall always take $C_c(X)$ to be equipped with this topology.

A **linear functional** on $C_c(X)$ is a linear function $\Phi : C_c(X) \to \mathbb{C}$. A functional is **positive** if the positive functions are taken to non-negative real numbers. Positive linear functionals are always continuous; see Exercise 2.8. If X is additionally compact, a linear functional Φ is called **normalized** if $\Phi(1_X) = 1$, where 1_X is the indicator function for X.

When X is a Polish group G, the group G has a left and a right action on $C_c(G)$. The left action is given by $L_g(f)(x) := f(g^{-1}x)$, and the right action is given by $R_g(f)(x) := f(xg)$. One verifies if $(f_i)_{i\in\mathbb{N}}$ converges to some $f \in C_c(G)$, then $L_g(f_i) \to L_g(f)$ and $R_g(f_i) \to R_g(f)$ for any $g \in G$; see Exercise 2.7. A linear functional Φ is **left-invariant** if $\Phi(L_g(f)) = \Phi(f)$ for all $f \in C_c(G)$ and $g \in G$, and it is **right-invariant** if $\Phi(R_g(f)) = \Phi(f)$ for all $f \in C_c(G)$ and $g \in G$.

A **sigma algebra** is a collection $\mathcal{S}$ of subsets of X such that $\mathcal{S}$ contains the empty set and is closed under taking complements, countable unions, and countable intersections. The **Borel sigma algebra** of X is the smallest sigma algebra that contains the open sets of X. The Borel sigma algebra is denoted by $\mathcal{B}(X)$ or just $\mathcal{B}$ when the space X is clear from context.

A measure μ defined on X is called a **Borel measure** if μ is defined on a sigma algebra $\mathcal{C}$ that contains the Borel sigma algebra of X. In practice, the natural choice of $\mathcal{C}$ is the sigma algebra generated by $\mathcal{B}$ and all null sets; a **null set** is a subset of a Borel set with measure zero. A Borel measure is said to be **locally finite** if for every $x \in X$ there is a neighborhood U of x with finite measure. A **probability measure** is a measure such that the measure of the entire space is 1.

The measures we construct on t.d.l.c. Polish groups shall have two important properties.

Definition 2.1. A locally finite Borel measure μ on a Polish space X is called an **outer Radon measure** if the following hold:

(i) (Outer regularity) For all $A \in \mathcal{B}(X)$,

$$\mu(A) = \inf\{\mu(U) \mid U \supseteq A \text{ is open}\}.$$

(ii) (Inner regularity) For all $A \in \mathcal{B}(X)$ such that $\mu(A) < \infty$,

$$\mu(A) = \sup\{\mu(K) \mid K \subseteq A \text{ is compact}\}.$$

We stress that by definition, outer Radon measures have good approximation properties. That is, the measure of a set can be approximated from above by open sets and from below by compact sets.

Definition 2.2. A Borel measure μ on a Polish group G is **left-invariant** if $\mu(gA) = \mu(A)$ for all Borel sets $A \subseteq G$ and $g \in G$. It is **right-invariant** if $\mu(Ag) = \mu(A)$ for all Borel sets $A \subseteq G$ and $g \in G$.

The classical Riesz's representation theorem relates measures on X to linear functionals on $C_c(X)$, when X is a locally compact Polish space.

Fact 2.3 (Riesz's representation theorem)**.** *Let X be a locally compact Polish space. If $\Phi : C_c(X) \to \mathbb{C}$ is a positive linear functional, then there exists a unique outer Radon measure μ such that*

$$\Phi(f) = \int_X f d\mu$$

for every $f \in C_c(X)$. If in addition X is compact and Φ is normalized, then μ is a probability measure.

2.2 Existence

We are now ready to prove the existence of left Haar measures.

Definition 2.4. For G a t.d.l.c. Polish group, a **left Haar measure** on G is a non-zero left-invariant outer Radon measure on G. The integral with respect to a left Haar measure is called the **left Haar integral**.

Theorem 2.5 (Haar)**.** *Every t.d.l.c. Polish group admits a left Haar measure.*

Our strategy to prove Theorem 2.5 will be to first consider compact t.d.l.c. Polish groups and then upgrade to the general case.

Remark 2.6. The proofs of this section can be adapted to produce a non-zero right-invariant outer Radon measure on G, which is called a **right Haar measure**. However, one cannot in general produce a non-zero outer Radon measure that is simultaneously left and right invariant.

2.2.1 The compact case

For the compact case of Theorem 2.5, we obtain a stronger result.

Theorem 2.7. *Every compact t.d.l.c. Polish group admits an outer Radon probability measure that is both left and right invariant.*

Our proof relies on the next lemma, Lemma 2.8, but let us assume it for the moment and prove the theorem.

Lemma 2.8. *For G a compact t.d.l.c. Polish group, there is a normalized positive linear functional Φ on $C(G)$ such that $\Phi(L_g(f)) = \Phi(f) = \Phi(R_g(f))$ for all $g \in G$ and $f \in C(G)$.*

Proof of Theorem 2.7. Let Φ be the normalized positive linear functional given by Lemma 2.8. Via Riesz's representation theorem, Φ defines an outer Radon probability measure μ on G. We argue that this measure is both left and right invariant.

As the proofs are the same, we argue that $\mu(gA) = \mu(A)$ for all $g \in G$ and $A \subseteq G$ Borel. Let us assume first that A is compact. Since the topology for G is given by cosets of clopen subgroups and A is compact, for any open set O with $A \subseteq O$, there is a clopen set O' such that $A \subseteq O' \subseteq O$. In particular, $\mu(O') \leq \mu(O)$. The outer regularity of μ now ensures that

$$\mu(A) = \inf\{\mu(O) \mid A \subseteq O \text{ with } O \text{ clopen}\}.$$

For any clopen set $O \subseteq G$, the indicator function 1_O is an element of $C(G)$. The left invariance of Φ ensures that

$$\mu(O) = \int_G L_g(1_O) d\mu = \int_G 1_{gO} d\mu = \mu(gO).$$

Since $\{gO \mid A \subset O \text{ and } O \text{ clopen}\}$ is exactly the collection of clopen sets containing gA, we deduce that

$$\begin{aligned} \mu(A) &= \inf\{\mu(O) \mid A \subseteq O \text{ with } O \text{ clopen}\} \\ &= \inf\{\mu(gO) \mid A \subseteq O \text{ with } O \text{ clopen}\} \\ &= \inf\{\mu(W) \mid gA \subseteq W \text{ with } W \text{ clopen}\} \\ &= \mu(gA). \end{aligned}$$

Hence, $\mu(A) = \mu(gA)$, so $\mu(A) = \mu(gA)$ for any $g \in G$ and $A \subseteq G$ compact.

Let us now consider an arbitrary Borel set A of G. By inner regularity,

$$\mu(A) = \sup\{\mu(K) \mid K \subseteq A \text{ with } K \text{ compact}\}.$$

Since $\{gK \mid K \subseteq A \text{ with } K \text{ compact}\}$ is exactly the collection of compact sets contained in gA, we conclude that

$$\begin{aligned} \mu(A) &= \sup\{\mu(K) \mid K \subseteq A \text{ with } K \text{ compact}\} \\ &= \sup\{\mu(gK) \mid K \subseteq A \text{ with } K \text{ compact}\} \\ &= \sup\{\mu(W) \mid W \subseteq gA \text{ with } W \text{ compact}\} \\ &= \mu(gA). \end{aligned}$$

Hence, $\mu(A) = \mu(gA)$ for any $g \in G$ and $A \subseteq G$ Borel. The measure μ thus satisfies Theorem 2.7. □

We now set about proving Lemma 2.8. The strategy will be to build the desired functional Φ via approximations. More precisely, we will define a sequence of subspaces $\mathcal{F}_n \subseteq C(G)$ and a sequence of functionals $\varphi_n : \mathcal{F}_n \to \mathbb{C}$ which approximate the desired behavior of Φ. These functionals will then limit to the desired functional Φ.

Fix G a compact t.d.l.c. Polish group. In view of Exercise 1.13, G admits a basis at 1 of compact open *normal* subgroups. Let $(G_n)_{n\in\mathbb{N}}$ be such a basis which is also $\subseteq$-decreasing. For each $n \in \mathbb{N}$, let $\mathcal{F}_n$ be the following vector subspace of $C(G)$:

$$\mathcal{F}_n := \{f \in \mathcal{C}(G) \mid \forall x \in G \text{ and } g \in G_n\ f(xg) = f(x)\}.$$

The subspace $\mathcal{F}_n$ consists of all continuous functions from G to $\mathbb{C}$ which are right G_n-invariant; alternatively, it is the collection of functions which are constant on *left* G_n cosets.

The subspaces $\mathcal{F}_n$ are setwise stabilized by G under both the left and the right actions. That is to say, $R_g(\mathcal{F}_n) = \mathcal{F}_n = L_g(\mathcal{F}_n)$ for all $g \in G$. We verify the former equality, as the latter is similar but easier.

Take $f \in \mathcal{F}_n$ and $g \in G$ and suppose that $x^{-1}y \in G_n$. We have that $R_g(f)(x) = f(xg)$ and $R_g(f)(y) = f(yg)$. Additionally, $(xg)^{-1}yg = g^{-1}x^{-1}yg$, and since G_n is a normal subgroup, $g^{-1}x^{-1}yg \in G_n$. Therefore, $f(xg) = f(yg)$, and so $R_g(f) \in \mathcal{F}_n$. It now follows that $R_g(\mathcal{F}_n) = \mathcal{F}_n$.

Fixing left coset representatives $b_1, \dots, b_r$ for G_n in G, we obtain a positive linear functional φ_n on $\mathcal{F}_n$ defined by

$$\varphi_n(f) := \sum_{i=1}^{r} \frac{f(b_i)}{|G : G_n|}.$$

The linear functional φ_n is independent of our choice of coset representatives, because the functions in $\mathcal{F}_n$ are constant on left G_n cosets.

Lemma 2.9. *For any $f \in \mathcal{F}_n$ and $g \in G$, $\varphi_n(L_g(f)) = \varphi_n(f) = \varphi_n(R_g(f))$.*

Proof. We only establish $\varphi_n(f) = \varphi_n(R_g(f))$; the other equality is similar but easier. Let $b_1, \ldots, b_r$ list coset representatives of G_n in G. Fix $g \in G$ and consider the set $b_i G_n g$. Since G_n is normal, we have $b_i g g^{-1} G_n g = b_i g G_n$. There is thus some j such that $b_i G_n g = b_j G_n$. We conclude that there is a permutation σ of $\{1, \ldots, r\}$ such that $b_i G_n g = b_{\sigma(i)} G_n$. Hence, $b_i g = b_{\sigma(i)} u_i$ for some $u \in G_n$. We now see that

$$\begin{aligned} \varphi_n(R_g(f)) &= \textstyle\sum_{i=1}^r \frac{f(b_i g)}{|G:G_n|} \\ &= \textstyle\sum_{i=1}^r \frac{f(b_{\sigma(i)} u)}{|G:G_n|} \\ &= \textstyle\sum_{i=1}^r \frac{f(b_{\sigma(i)})}{|G:G_n|} \\ &= \varphi(f), \end{aligned}$$

where the penultimate line follows since f is constant on left cosets of G_n. $\square$

Our next lemma shows the sequence $(\varphi_n)_{n \in \mathbb{N}}$ has a coherence property, which will allow us to take a limit.

Lemma 2.10. *For every $n \leq m$, the vector space $\mathcal{F}_n$ is a subspace of $\mathcal{F}_m$, and φ_n is the restriction of φ_m to $\mathcal{F}_n$.*

Proof. Take $f \in \mathcal{F}_n$. For any $x \in G$ and $g \in G_n$, we have $f(x) = f(xg)$. On the other hand, $G_n \geq G_m$, so a fortiori, $f(x) = f(xg)$ for any $x \in G$ and $g \in G_m$. Hence, $f \in \mathcal{F}_m$, and we deduce that $\mathcal{F}_n \leq \mathcal{F}_m$.

Let us now verify that $\varphi_m \restriction_{\mathcal{F}_n} = \varphi_n$. We see that

$$|G : G_n||G_n : G_m| = |G : G_m|,$$

so

$$\frac{|G_n : G_m|}{|G : G_m|} = \frac{1}{|G : G_n|}.$$

Setting $k := |G_n : G_m|$ and $r := |G : G_n|$, let $a_1, \ldots, a_k$ be left coset representatives for G_m in G_n and let $b_1, \ldots, b_r$ be left coset representatives for G_n in G. Fixing $f \in \mathcal{F}_n$,

$$\varphi_m(f) = \sum_{i=1}^r \sum_{j=1}^k \frac{f(b_i a_j)}{|G : G_m|}.$$

The function f is constant on b_iG_n, so

$$\varphi_m(f) = \sum_{i=1}^{r} \frac{f(b_i a_1)k}{|G : G_m|} = \sum_{i=1}^{r} \frac{f(b_i)}{|G : G_n|}$$

Hence, $\varphi_m(f) = \varphi_n(f)$. □

We now argue the subspaces $\mathcal{F}_n$ essentially exhaust $C(G)$. Let $\mathcal{F}_n^+$ be the positive functions in $\mathcal{F}_n$.

Lemma 2.11. *The union $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ is dense in $C(G)$, and $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n^+$ is dense in $C(G)^+$.*

Proof. Fix $\epsilon > 0$ and $f \in C(G)$. For every $x \in G$, the continuity of f ensures that there is an element G_i of our basis at the identity such that $\mathrm{diam}(f(xG_i)) < \epsilon$. Since G is compact and zero dimensional, we may find $x_1, \dots, x_p$ elements of G and natural numbers $n_1, \dots, n_p$ such that $(x_iG_{n_i})_{i=1}^p$ is a covering of G by disjoint open sets.

Fix $N \in \mathbb{N}$ such that

$$G_N \leq \bigcap_{i=1}^{p} G_{n_i},$$

and fix $r_x \in f(xG_N)$ for each left G_N-coset xG_N. We define $\tilde{f} \in \mathcal{F}_N$ by $\tilde{f}(x) := r_x$; that is, $\tilde{f}$ is the function that takes value r_x for any $y \in xG_N$. For $x \in G$, there is some $1 \leq i \leq p$ with $x \in x_iG_{n_i}$, so $xG_{n_i} = x_iG_{n_i}$. We infer that $f(x) \in f(x_iG_{n_i})$. On the other hand, xG_N is a subset of $x_iG_{n_i}$, as $G_N \leq G_{n_i}$, so $\tilde{f}(x) = r_x \in f(x_iG_{n_i})$. We now see that $|r_x - f(x)| < \epsilon$. Therefore, $\|\tilde{f} - f\| < \epsilon$, and we conclude that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ is dense in $C(G)$.

The second claim follows since if f positive, we can pick $r_x \geq 0$ for each coset $xG_N \in G/G_N$. □

In view of Lemma 2.10, $\varphi_n(f) = \varphi_m(f)$ whenever $f \in \mathcal{F}_n \cap \mathcal{F}_m$. We may thus define a linear functional φ on $\mathcal{F} := \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ by $\varphi(f) := \varphi_n(f)$ for $f \in \mathcal{F}_n$. The linear functional φ is positive and normalized, and Lemma 2.11 tells us the domain of φ is dense in $C(X)$.

We now argue that we may extend φ to $C(X)$. To avoid appealing to unproven facts about positive linear functionals, we verify that φ carries Cauchy sequences to Cauchy sequences.

Lemma 2.12. *If $(f_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in $C(X)$ with $f_i \in \mathcal{F}$ for all $i \in \mathbb{N}$, then $(\varphi(f_i))_{i\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$.*

Proof. Fix $\epsilon > 0$ and let N be such that $\|f_i - f_j\| < \epsilon$ for all $i, j \geq N$. Fix $i, j \geq N$ and let k be such that $f_i, f_j \in \mathcal{F}_k$. Letting $b_1, \ldots, b_m$ list left coset representatives for G_k in G,

$$\begin{array}{rcl} |\varphi(f_i) - \varphi(f_j)| & = & |\varphi_k(f_i) - \varphi_k(f_j)| \\ & = & |\sum_{l=1}^{m} \frac{f_i(b_l) - f_j(b_l)}{|G:G_k|}| \\ & \leq & \sum_{l=1}^{m} \frac{|f_i(b_l) - f_j(b_l)|}{|G:G_k|} \\ & < & \sum_{l=1}^{m} \frac{\epsilon}{|G:G_k|} \\ & = & \epsilon. \end{array}$$

We conclude that $(\varphi(f_i))_{i \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$. □

Lemmas 2.11 and 2.12 allow us to extend φ to a linear functional Φ on $C(G)$. For $f \in \mathcal{F}$, we define $\Phi(f) := \phi(f)$. For $f \in C(G) \setminus \mathcal{F}$, Lemma 2.11 supplies a sequence $(f_i)_{i \in \mathbb{N}}$ from $\mathcal{F}$ that converges to f. We then define $\Phi(f) := \lim_{i \in \mathbb{N}} \varphi(f_i)$, and this limit exists by Lemma 2.12. Note that the definition of Φ does not depend on the choice of the sequence $(f_i)_{i \in \mathbb{N}}$; see Exercise 2.1.

We now argue that Φ satisfies Lemma 2.8. It is immediate that Φ is normalized since $1_X \in \mathcal{F}$. To see that Φ is positive, let $f \in C(G)$ be a positive function. By Lemma 2.11, there is a sequence of positive functions $(f_i)_{i \in \mathbb{N}}$ with $f_i \in \mathcal{F}$ which converges to f, so

$$\Phi(f) = \lim_{i \in \mathbb{N}} \varphi(f_i) = \lim_{i \in \mathbb{N}} \varphi_i(f_i) \geq 0.$$

The functional Φ is thus also positive.

Finally, let us verify that Φ is invariant under the left and right actions of G. Take $g \in G$ and $f \in C(X)$. If $f \in \mathcal{F}$, then

$$\Phi(L_g(f)) = \varphi(L_g(f)) = \varphi(f) = \varphi(R_g(f)) = \Phi(R_g(f))$$

by Lemma 2.9. Suppose that $f \in C(X) \setminus \mathcal{F}$ and fix $f_i \to f$ a convergent sequence such that $f_i \in \mathcal{F}$ for all i. Recall that $L_g(f_i) \to L_g(f)$ and $R_g(f_i) \to R_g(f)$. Applying the definition of Φ and Lemma 2.9, we deduce that

$$\Phi(L_g(f)) = \lim_{i \in \mathbb{N}} \varphi(L_g(f_i)) = \lim_{i \in \mathbb{N}} \varphi(f_i) = \lim_{i \in \mathbb{N}} \varphi(R_g(f_i)) = \Phi(R_g(f)),$$

hence $\Phi(L_g(f)) = \Phi(f) = \Phi(R_g(f))$.

We have now proved Lemma 2.8.

2.2.2 The general case

The general case now follows by a clever trick, which allows one to glue together measures.

Lemma 2.13. *Suppose that X is a Polish space and X is a disjoint union of clopen sets U_i. If each U_i admits an outer Radon probability measure ν_i, then the function $\mu : \mathcal{B} \to [0, \infty]$ by*

$$\mu(B) := \sum_{i=1}^{\infty} \nu_i(B \cap U_i),$$

is a non-trivial outer Radon measure on X.

Proof. That μ is a measure, we leave to the reader; see Exercise 2.4. To see that μ is locally finite, for any $x \in X$, there is U_i such that $x \in U_i$. Hence, $\mu(U_i) = \nu_i(U_i) = 1$, so μ is locally finite.

Fix $B \in \mathcal{B}$. Outer regularity is immediate from the definition of μ if $\mu(B) = \infty$. Let us suppose that $\mu(B) < \infty$ and fix $\epsilon > 0$. For each $i \geq 1$, the outer regularity of ν_i supplies an open set $W_i \subseteq U_i$ such that $B \cap U_i \subseteq W_i$ and $\nu_i(W_i) - \nu_i(B \cap U_i) < \frac{\epsilon}{2^i}$. The set $W := \bigcup_{i=1}^{\infty} W_i$ is open in X. Furthermore,

$$\begin{array}{rcl} \mu(W) & = & \sum_{i=1}^{\infty} \nu_i(W_i) \\ & < & \sum_{i=1}^{\infty} \left(\nu_i(B \cap U_i) + \frac{\epsilon}{2^i}\right) \\ & = & \mu(B) + \epsilon. \end{array}$$

Since $B \subseteq W$, we deduce that $|\mu(W) - \mu(B)| < \epsilon$. The measure μ is thus outer regular.

For inner regularity, we may assume $\mu(B) < \infty$. Fix $\epsilon > 0$. Since the series $\sum_{i=1}^{\infty} \nu_i(B \cap U_i)$ converges, we may find N such that $\sum_{i=N+1}^{\infty} \nu_i(B \cap U_i) < \frac{\epsilon}{2}$. For each $1 \leq i \leq N$, there is a compact set $K_i \leq B \cap U_i$ such that $\nu_i(B \cap U_i) - \mu_i(K_i) < \frac{\epsilon}{2N}$, by the inner regularity of ν_i. The set $K := \bigcup_{i=1}^{N} K_i$ is compact in X, and furthermore,

$$\begin{array}{rcl} \mu(B) & = & \sum_{i=1}^{\infty} \nu_i(B \cap U_i) \\ & < & \left(\sum_{i=1}^{N} \nu_i(K_i) + \frac{\epsilon}{2N}\right) + \sum_{i=N+1}^{\infty} \nu_i(B \cap U_i) \\ & = & \mu(K) + \epsilon. \end{array}$$

Since $K \subseteq B$, we deduce that $|\mu(B) - \mu(K)| < \epsilon$. The measure μ is thus inner regular. $\square$

For topological spaces X and Y, a function $f : X \to Y$ is **Borel measurable** if $f^{-1}(O)$ is in the Borel sigma algebra of X for any O open in Y. If X is additionally equipped with a Borel measure μ, then the **push forward** measure on Y via f, denoted by $f_*\mu$, is defined by

$$f_*\mu(A) := \mu(f^{-1}(A)).$$

If f is a homeomorphism and μ is an outer Radon measure, then $f_*\mu$ is also an outer Radon measure.

We are now prepared to prove the general case.

Theorem 2.14 (Haar)**.** *Every t.d.l.c. Polish group admits a left Haar measure.*

Proof. Let G be a t.d.l.c. Polish group and fix a compact open subgroup U of G. If U is of finite index in G, then G is a compact t.d.l.c. Polish group, so Theorem 2.7 supplies a left Haar measure on G. We thus suppose that U is of infinite index. Take $(g_i)_{i\in\mathbb{N}}$ a sequence of left coset representatives such that G is the disjoint union of $(g_iU)_{i\in\mathbb{N}}$.

Since U is a compact t.d.l.c. Polish group, Theorem 2.7 supplies a outer Radon probability measure ν on U which is both left and right invariant. Equip each g_iU with the pushforward measure $\nu_i := g_{i*}\nu$. That is to say, for $B \subseteq g_iU$ Borel,

$$\nu_i(B) = \nu(g_i^{-1}(B)).$$

The measure ν_i is an outer Radon measure on g_iU, since $g_i : U \to g_iU$ is a homeomorphism.

Applying Lemma 2.13, we obtain a non-trivial outer Radon measure μ on G such that for $B \subseteq G$ Borel,

$$\mu(B) := \sum_{i\in\mathbb{N}} \nu_i(B \cap g_iU).$$

Unpacking the definitions reveals that

$$\mu(B) = \sum_{i\in\mathbb{N}} \nu(g_i^{-1}(B \cap g_iU)) = \sum_{i\in\mathbb{N}} \nu(g_i^{-1}(B) \cap U).$$

We now argue that μ is left-invariant. For $g \in G$, there is a permutation σ of $\mathbb{N}$ such that $gg_iU = g_{\sigma(i)}U$, so in particular, $g_{\sigma(i)}^{-1}gg_i$ is in U. For all Borel

$B \subseteq X$, we now see that

$$\begin{aligned}\mu(g^{-1}(B)) &= \textstyle\sum_{i\in\mathbb{N}} \nu((gg_i)^{-1}(B)\cap U) \\ &= \textstyle\sum_{i\in\mathbb{N}} \nu\left(g_{\sigma(i)}^{-1} g g_i\left((gg_i)^{-1}(B)\cap U\right)\right) \\ &= \textstyle\sum_{i\in\mathbb{N}} \nu(g_{\sigma(i)}^{-1}(B)\cap U) \\ &= \mu(B).\end{aligned}$$

The second line follows from the left U-invariance of ν. We conclude that μ is left-invariant, and so μ is a left Haar measure on G. □

2.3 Uniqueness

The previous section establishes that Haar measures exist, but our construction requires many choices. One thus wishes to know if the Haar measure is unique in some way, and we here find and prove such a uniqueness property.

Proving the desired theorem requires additional preliminary results.

Lemma 2.15. *Let μ be a left Haar measure on a non-trivial t.d.l.c. Polish group G.*

(1) Every non-empty open set has strictly positive (possibly infinite) measure.

(2) Every compact set has finite measure.

(3) Every continuous positive function $f : G \to \mathbb{R}$ with $\int_G f d\mu = 0$ vanishes identically.

Proof. Exercise 2.15 □

Integration with respect to the left Haar measure enjoys a left invariance property, which follows easily from the left invariance of the left Haar measure. For a measure space (X, μ), the collection of integrable functions $f : X \to \mathbb{C}$ is denoted by $L^1(X, \mu)$; when the space X is clear from context, we simply write $L^1(\mu)$.

Lemma 2.16. *Let μ be a left Haar measure on a t.d.l.c. Polish group G. For any $g \in G$ and $f \in L^1(\mu)$,*

$$\int_G L_g(f) d\mu = \int_G f d\mu.$$

Proof. Exercise 2.17. □

We indeed only apply Lemma 2.16 in the case of $f \in C_c(G)$, and this case follows by approximating a given $f \in C_c(G)$ by functions constant on left cosets of some compact open subgroup; see Exercise 2.18.

As every analysis student knows, continuous functions on compact spaces are uniformly continuous. We upgrade this to our compactly supported functions on a locally compact group. The group structure allows us to avoid using a metric on the locally compact group.

Lemma 2.17. *Let G be a t.d.l.c. Polish group and $f \in C_c(G)$. For every $\epsilon > 0$, there is $U \in \mathcal{U}(G)$ such that if $x^{-1}y \in U$ or $yx^{-1} \in U$, then $|f(x) - f(y)| < \epsilon$. That is to say, f is uniformly continuous.*

Proof. Fix $\epsilon > 0$. We will find $U \in \mathcal{U}(G)$ such that if $x^{-1}y \in U$, then $|f(x) - f(y)| < \epsilon$; the other case is similar. One then simply intersects the compact open subgroups obtained in each argument to find a compact open subgroup that satisfies the lemma.

Fix $f \in C_c(G)$, let $U \in \mathcal{U}(G)$, and let $K := \overline{\mathrm{supp}(f)}$. For each $x \in G$, the continuity of f supplies a compact open subgroup $V_x \leq U$ such that if $y \in xV_x$, then $|f(x) - f(y)| < \frac{\epsilon}{2}$. Since KU is compact, there are $x_1, \cdots, x_n$ such that $KU = x_1V_{x_1} \cup \cdots \cup x_nV_{x_n}$. We claim $W := \bigcap_{i=1}^n V_{x_i}$ satisfies the lemma.

Suppose $x^{-1}y \in W$. If $x \notin KU$, then $y \notin K$. (Consider the converse.) Thus, $f(x) = f(y) = 0$, and the desired inequality clearly holds in this case. Suppose $x \in KU$; say $x \in x_iV_{x_i}$. It is then the case that $y \in x_iV_{x_i}$, so

$$|f(x) - f(y)| \leq |f(x) - f(x_i)| + |f(x_i) - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2}.$$

proving the lemma. □

We require one last and rather technical lemma.

Lemma 2.18. *Let μ be a left Haar measure on G. For $f \in C_c(G)$, the function $\Psi_f : G \to \mathbb{C}$ defined by*

$$s \mapsto \int_G f(xs)d\mu(x)$$

is continuous.

Proof. Let us first show that Ψ_f is continuous at 1 for every $f \in C_c(G)$. That is, for every $\epsilon > 0$ there is $U \in \mathcal{U}(G)$ such that

$$\left| \int_G f(xs) - f(x) d\mu(x) \right| < \epsilon$$

for every $s \in U$.

Let $K := \operatorname{supp}(f)$, fix $\epsilon > 0$, and take $V \in \mathcal{U}(G)$. For $s \in V$, we see $\operatorname{supp}(R_s(f)) \subseteq KV$, and in view of Lemma 2.15, we observe that $0 < \mu(KV) < \infty$. Lemma 2.17 now supplies a compact open subgroup $W \leq V$ such that for all $s \in W$

$$|f(xs) - f(x)| < \frac{\epsilon}{\mu(KV)}.$$

Therefore,

$$\left| \int_G f(xs) - f(x) d\mu(x) \right| \leq \int_{KV} |f(xs) - f(x)| d\mu(x) \leq \frac{\epsilon}{\mu(KV)} \mu(KV).$$

verifying that Ψ_f is continuous at 1.

We now argue that Ψ_f is in fact continuous. Take O open in $\mathbb{C}$ and fix r in $\Psi_f^{-1}(O)$. The function $R_r(f)$ is again an element of $C_c(G)$ and $\Psi_{R_r(f)}(1) \in O$. The previous paragraph now ensures that $(\Psi_{R_r(f)})^{-1}(O)$ contains L an open neighborhood of 1 such that $\Psi_{R_r(f)}(L) \subseteq O$. For any $zr \in Lr$,

$$\Psi_f(zr) = \int_G f(xzr) d\mu(x) = \int_G R_r(f(xz)) d\mu(x) = \Psi_{R_r(f)}(z),$$

so $\Psi_f(Lr) \subseteq O$, verifying that Ψ_f is continuous. $\square$

Our proof finally requires the classical Fubini–Tonelli theorem. A measure space (X, μ) is called **sigma finite** if $X = \bigcup_{n \in \mathbb{N}} W_n$ such that each W_n is measurable with $\mu(W_n) < \infty$. Any Haar measure is sigma finite by Exercise 2.10.

Fact 2.19 (Fubini–Tonelli)**.** *Suppose that (X, μ) and (Y, ν) are sigma finite measure spaces and take $f \in L^1(\mu \times \nu)$, the space of integrable functions $f : X \times Y \to \mathbb{C}$ where $X \times Y$ is equipped with the product measure. Then,*

$$\begin{aligned} \textstyle\int_{X \times Y} f(x, y) d(\mu \times \nu) &= \textstyle\int_X \left(\int_Y f(x, y) d\nu(y) \right) d\mu(x) \\ &= \textstyle\int_Y \left(\int_X f(x, y) d\mu(x) \right) d\nu(y). \end{aligned}$$

We now state and prove the uniqueness theorem for Haar measure.

Theorem 2.20 (Haar). *A left Haar measure on a t.d.l.c. Polish group is unique up to constant multiplies. That is to say, for any two left Haar measures ν and μ on a t.d.l.c. Polish group G, there is a non-zero real number c such that $\nu = c\mu$.*

Proof. For $h \in C_c(G)$, set $I_\mu(h) := \int_G h d\mu$ and $I_\nu(h) := \int_G h d\nu$. Fix $f \in C_c(G)$ such that $I_\mu(f) \neq 0$ and define

$$D_f(s) := \frac{1}{I_\mu(f)} \int_G f(ts) d\nu(t).$$

By Lemma 2.18, $D_f(s)$ is a continuous function of s.

Take $g \in C_c(G)$. Via the Fubini–Tonelli theorem,

$$I_\mu(f) I_\nu(g) = \int_G \left(\int_G f(s) g(t) d\nu(t) \right) d\mu(s).$$

Lemma 2.16 ensures that

$$\begin{aligned} \int_G \int_G f(s) g(t) d\nu(t) d\mu(s) &= \int_G \int_G L_s(f(s) g(t)) d\nu(t) d\mu(s) \\ &= \int_G \int_G f(s) g(s^{-1}t) d\nu(t) d\mu(s). \end{aligned}$$

A second application of the Fubini–Tonelli theorem yields

$$\int_G \int_G f(s) g(s^{-1}t) d\nu(t) d\mu(s) = \int_G \left(\int_G f(s) g(s^{-1}t) d\mu(s) \right) d\nu(t),$$

and appealing again to invariance as well as the Fubini–Tonelli theorem,

$$\begin{aligned} \int_G \int_G f(s) g(s^{-1}t) d\mu(s) d\nu(t) &= \int_G \int_G L_{t^{-1}}(f(s) g(s^{-1}t)) d\mu(s) d\nu(t) \\ &= \int_G \int_G f(ts) g(s^{-1}) d\nu(t) d\mu(s) \\ &= I_\mu(f) \int_G D_f(s) g(s^{-1}) d\mu(s). \end{aligned}$$

We have now demonstrated that

$$I_\mu(f) I_\nu(g) = I_\mu(f) \int_G D_f(s) g(s^{-1}) d\mu(s),$$

and thus, $I_\nu(g) = \int_G D_f(s) g(s^{-1}) d\mu(s)$.

For any other function $f' \in C_c(G)$ with $I_\mu(f') \neq 0$, it is thus the case that

$$\int_G \left(D_f(s) - D_{f'}(s) \right) g(s^{-1}) d\mu(s) = 0.$$

As this equality holds for any $g \in C_c(G)$, let us replace g with

$$\tilde{g}(s) := |g(s)|^2 \overline{(D_f(s^{-1}) - D_{f'}(s^{-1}))},$$

where $\overline{(D_f(s^{-1}) - D_{f'}(s^{-1}))}$ denotes the complex conjugate. Observe that $\tilde{g} \in C_c(G)$, via Lemma 2.18. We now see that

$$\begin{array}{rcl} 0 & = & \int_G (D_f(s) - D_{f'}(s))\, \tilde{g}(s^{-1}) d\mu(s) \\ & = & \int_G |D_f(s) - D_{f'}(s)|^2 |g(s^{-1})|^2 d\mu(s), \end{array}$$

and in view of Lemma 2.15, $(D_f(s) - D_{f'}(s))g(s^{-1}) = 0$ for all s. Since $g(x)$ is arbitrary, we conclude that $D_f(s) = D'_f(s)$ for all f and f' in $C_c(G)$ with non-zero integral, and hence, $D_f(s) = c$ for some constant c independent of f.

Recalling the definition of $D_f(s)$, we see that

$$c = D_f(s) = \frac{1}{I_\mu(f)} \int_G f(ts) d\nu(t).$$

Taking $s = 1$, we deduce that

$$cI_\mu(f) = \int_G f(t) d\nu(t) = I_\nu(f).$$

It now follows that $c\mu = \nu$, and therefore, the Haar measure is unique up to constant multiples; see Exercise 2.13. □

2.4 The modular function

The uniqueness of the Haar measure has many consequences. One of the first is the existence of a special homomorphism, called the modular function.

Let G be a t.d.l.c. Polish group with left Haar measure μ and $x \in G$. We may produce a new left Haar measure μ_x by defining $\mu_x(A) := \mu(Ax)$. The uniqueness of the Haar measure implies there is $\Delta(x) > 0$ such that $\mu_x = \Delta(x)\mu$.

Definition 2.21. The map $\Delta : G \to \mathbb{R}_{>0}$ defined above is called the **modular function** for G. If $\Delta \equiv 1$, then G is said to be **unimodular**.

It may seem the modular function depends on the choice of Haar measure μ, but this is not the case.

Lemma 2.22. *Let G be a t.d.l.c. Polish group with μ_1 and μ_2 left Haar measures on G. Taking Δ_1 and Δ_2 to be the modular functions defined in terms of μ_1 and μ_2, respectively, $\Delta_1 = \Delta_2$.*

Proof. Since the Haar measure is unique up to constant multiples, there is a positive real number D such that $\mu_1 = D\mu_2$. For all $x \in G$ and $A \subseteq G$ measurable, we have

$$D\mu_2(Ax) = \mu_1(Ax) = \Delta_1(x)\mu_1(A) = \Delta_1(x)D\mu_2(A).$$

Hence, $\mu_2(Ax) = \Delta_1(x)\mu_2(A)$ for all $x \in G$ and $A \subseteq G$ measurable, so $\Delta_2 = \Delta_1$. $\square$

We now show the modular function is a continuous homomorphism.

Proposition 2.23. *For G a t.d.l.c. Polish group, the modular function $\Delta : G \to \mathbb{R}_{>0}$ is a continuous group homomorphism.*

Proof. We first argue Δ is a group homomorphism. Fixing $U \in \mathcal{U}(G)$, $x, y \in G$, and μ a left Haar measure,

$$\begin{aligned} \Delta(xy)\mu(U) &= \mu(Uxy) \\ &= \Delta(y)\mu(Ux) \\ &= \Delta(x)\Delta(y)\mu(U). \end{aligned}$$

Since $0 < \mu(U) < \infty$, $\Delta(xy) = \Delta(x)\Delta(y)$ verifying that Δ is a group homomorphism.

To see that Δ is continuous, we now have only to check continuity at 1. Fix W a compact open subgroup. By the uniqueness of Haar measure, μ restricted to W is a Haar measure on W. Theorem 2.7 therefore implies that $\mu(Aw) = \mu(A)$ for all measurable $A \subseteq W$ and $w \in W$. Hence, $\Delta(w) = 1$ for all $w \in W$, so Δ is continuous at 1. $\square$

Remark 2.24. For G a t.d.l.c. Polish group and $H \leq G$ a closed subgroup, the left Haar measure μ_H on H and the left Haar measure μ_G on G are often very different. For instance, unless H is open in G, $\mu_G(H) = 0$; see Exercise 2.22. The respective modular functions can also differ, so we will write Δ_H to denote the modular function of H and Δ_G to denote that of G.

We may now state a version of Lemma 2.16, but now for the right translation.

Lemma 2.25. *Let μ be a left Haar measure on a t.d.l.c. Polish group G. For any $g \in G$ and $f \in L^1(\mu)$,*

$$\int_G R_g(f)d\mu = \Delta(g^{-1}) \int_G f d\mu.$$

Proof. Exercise 2.17. □

The modular function also relates the integral of $f(x^{-1})$ with that of $f(x)$.

Lemma 2.26. *Let μ be a left Haar measure on a t.d.l.c. Polish group G. For any $f \in L^1(\mu)$,*

$$\int_G f(x)d\mu = \int_G \Delta(x^{-1})f(x^{-1})d\mu.$$

Proof. Set $I(f) := \int_G \Delta(x^{-1})f(x^{-1})d\mu$. By Lemma 2.16,

$$\begin{array}{rcl} I(L_z(f)) & = & \int_G (f(z^{-1}x^{-1})\Delta(x^{-1})d\mu \\ & = & \int_G f((xz)^{-1})\Delta(x^{-1})d\mu \\ & = & \Delta(z^{-1}) \int_G f(x^{-1})\Delta((xz^{-1})^{-1})d\mu \\ & = & \int_G f(x^{-1})\Delta(x^{-1})d\mu, \end{array}$$

where the last line follows since Δ is a homomorphism. We conclude that $I(L_z(f)) = I(f)$, and therefore, I is a left invariant positive linear functional.

Via the Riesz representation theorem, I is indeed integration against the left Haar measure. The uniqness of the left Haar measure ensures that $I(f) = c\int_G f(x)d\mu$ for some positive constant c. We argue that $c = 1$, which completes the proof.

Fix $\epsilon > 0$ and let U be a compact open subgroup such that $|1-\Delta(g)| < \epsilon$ for all $g \in U$. Fix $f \in C_c^+(G)$ such that $\mathrm{supp}(f) \subseteq U$, $f(x) = f(x^{-1})$, and $\int_G f(x)d\mu \neq 0$. We see that

$$\begin{array}{rcl} |1-c| \int_G f(x)d\mu & = & |\int_G f(x)d\mu - I(f)| \\ & \leq & \int_G |f(x) - f(x^{-1})\Delta(x^{-1})|d\mu \\ & = & \int_U f(x)|1-\Delta(x^{-1})|d\mu \\ & < & \epsilon \int_G f(x)d\mu. \end{array}$$

Hence, $|1-c| < \epsilon$, and since ϵ is arbitrary, it is the case that $c = 1$. □

2.5 Quotient integral formula

The last result of this chapter gives important insight into the relationship between the Haar measure on closed subgroup and the Haar measure of the ambient group.

We being by examining an important averaging operation. Suppose that G is a t.d.l.c. Polish group and H is a closed subgroup with left Haar measure μ_H. For $f \in C_c(G)$, define

$$f^H(x) := \int_H f(xh)d\mu_H(h).$$

The function f^H is well-defined since $h \mapsto f(xh)$ is a continuous function with compact support for any x, and therefore, $\int_H f(xh)d\mu_H(h)$ exists.

We will require two important observations about the operation of forming f^H.

Lemma 2.27. *Let G be a t.d.l.c. Polish group and H a closed subgroup with left Haar measure μ_H. For $f \in C_c(G)$, the function f^H is an element of $C_c(G/H)$ with* $\mathrm{supp}(f^H) \subseteq \mathrm{supp}(f)H/H$.

Proof. Let us first see that f^H is indeed well-defined on G/H. Fix $x \in G$ and let $\tilde{f} \in C_c(H)$ be defined by $\tilde{f}(h) := f(xh)$. We see that

$$\int_H \tilde{f}(h)d\mu_H(h) = f^H(x).$$

For $k \in H$, Lemma 2.16 ensures that

$$\int_H L_{k^{-1}}(\tilde{f}(h))d\mu_H(h) = \int_H \tilde{f}(h)d\mu_H(h),$$

so

$$f^H(x) = \int_H \tilde{f}(kh)d\mu_H(h) = \int_H f(xkh)d\mu_H(h) = f^H(xk).$$

The function $f^H(x)$ is thus constant on left H cosets, and f^H is well-defined on G/H.

It is easy to see that $\mathrm{supp}(f^H(x)) \subseteq \mathrm{supp}(f)H/H$, and hence, $f^H(x)$ is compactly supported; we leave the details as Exercise 2.19. Let us now argue that f^H is continuous. Fix $\epsilon > 0$, $y \in G$, and U a compact open subgroup of G. Set $K := \mathrm{supp}(f)$ and $d := \mu_H(y^{-1}UK \cap H)$. Applying Lemma 2.17, there is a compact open subgroup $V \leq U$ such that $|f(z) - f(w)| < \frac{\epsilon}{d}$ whenever $zw^{-1} \in V$. In particular, $|f(xh) - f(yh)| < \frac{\epsilon}{d}$ for any $h \in H$ and $x \in Vy$.

For any $x \in Vy$, observe that $f(xh)$ is non-zero only if $h \in y^{-1}UK \cap H$. Taking $x \in Vy$, we thus deduce that

$$\begin{array}{rcl} |f^H(x) - f^H(y)| & \leq & \int_H |f(xh) - f(yh)| d\mu_H(h) \\ & = & \int_{y^{-1}UK\cap H} |f(xh) - f(yh)| d\mu_H(h) \\ & < & \int_{y^{-1}UK\cap H} \frac{\epsilon}{d} d\mu_H(h) \\ & = & \epsilon \end{array}$$

Hence, f^H is continuous. □

Lemma 2.28. *Let G be a t.d.l.c. Polish group and H a closed subgroup with left Haar measure μ_H. The map $C_c(G) \to C_c(G/H)$ by $f \mapsto f^H$ is surjective.*

Proof. Lemma 2.27 ensures the map under consideration is well-defined.

Fix $g \in C_c(G/H)$, let $K \subseteq G/H$ be a compact set containing $\text{supp}(g)$, and let $\pi : G \to G/H$ be the usual projection. Fixing a compact open subgroup U of G, $\pi(kU)$ is an compact open neighborhood of $kH \in K$. Since K is compact, we may find a finite set $F \subseteq G$ such that $\pi(FU)$ is a compact open set containing K.

Since FU is clopen and compact, $\phi : G \to \mathbb{C}$ defined by

$$\phi(x) := \begin{cases} 1 & \text{if } x \in FU \\ 0 & \text{otherwise} \end{cases}$$

is continuous and compactly supported; that is to say, $\phi \in C_c(G)$. The function $f : G \to \mathbb{C}$ defined by

$$f(x) := \begin{cases} \frac{g\circ\pi(x)\cdot\phi(x)}{\phi^H(x)} & \text{if } x \in FU \\ 0 & \text{otherwise} \end{cases}$$

is thus also in $C_c(G)$ since $\text{supp}(g) \subseteq \pi(FU)$.

The function $\phi^H(x)$ is right H invariant, so it follows that

$$f^H(x) = \frac{g \circ \pi(x) \cdot \phi^H(x)}{\phi^H(x)} = g.$$

The map $f \mapsto f^H$ is therefore surjective. □

For a group G acting on a measure space (X, μ), we say that μ is **invariant** under the action of G if $\mu(A) = \mu(g(A))$ for all $g \in G$ and measurable $A \subseteq X$. Given a t.d.l.c. Polish group G and a closed subgroup $H \leq G$, there is a natural action of G on G/H by left multiplication. The next theorem characterizes when G/H admits an invariant measure in terms of the modular functions on H and G.

Theorem 2.29. *For G a t.d.l.c. Polish group and $H \leq G$ a closed subgroup, G/H admits a non-zero invariant outer Radon measure ν if and only if $\Delta_G \restriction_H = \Delta_H$.*

Proof. Suppose first that G/H admits such a measure ν and let μ_H be the left Haar measure on H. We define a positive linear functional I on $C_c(G)$ by

$$I(f) := \int_{G/H} f^H(xH)d\nu(xH).$$

In view of Riesz's representation theorem, there is a non-zero outer Radon measure μ_G on G such that $I(f) = \int_G f(x)d\mu_G(x)$. Since ν is invariant under the action of G on G/H, $I(f)$ is left invariant- i.e. $I(f) = I(L_g(f))$ for any $g \in G$; this follows as Lemma 2.16. Hence, μ_G is also left invariant and thus is the left Haar measure on G

Fix $h_0 \in H$. Let Δ_H be the modular function on H and Δ_G be the modular function on G. Take $f \in C_c(G)$. In view of Lemma 2.25, we see that

$$\begin{array}{rcl}\Delta_G(h_0)\int_G f(x)d\mu_G(x) & = & \int_G f(xh_0^{-1})d\mu_G(x) \\ & = & \int_G R_{h_0^{-1}}(f(x))d\mu_G(x).\end{array}$$

In view of the way by which we produced μ_G and applying again Lemma 2.25, we have

$$\begin{array}{rcl}\int_G R_{h_0^{-1}}(f(x))d\mu_G(x) & = & \int_{G/H}\int_H f(xhh_0^{-1})d\mu_H(h)d\nu(xH) \\ & = & \Delta_H(h_0)\int_{G/H}\int_H f(xh)d\mu_H(h)d\nu(xH) \\ & = & \Delta_H(h_0)\int_G f(x)d\mu_G(x).\end{array}$$

We can choose $f \in C_c(G)$ with a non-zero integral, so we conclude that $\Delta_G(h_0) = \Delta_H(h_0)$. Hence, $\Delta_G \restriction_H = \Delta_H$.

Conversely, suppose that $\Delta_G \restriction_H = \Delta_H$. The map $\Psi : C_c(G) \to C_c(G/H)$ by $f \mapsto f^H$ is surjective by Lemma 2.28. One verifies, see Exercise 2.19, that Ψ is additionally continuous and linear. Letting K be the kernel and $\tilde{\Psi}$ be the induced map on $C_c(G)/K$, we see that $\tilde{\Psi} : C_c(G)/K \simeq C_c(G/H)$ as vector spaces.

Fixing a left Haar measure μ_G on G, we argue that the Haar integral is well-defined on $C_c(G)/K$. Fix $f \in K$ and let $g \in C_c(G)$. Since $f \in K$,

$$0 = \int_G \int_H f(xh)g(x)d\mu_H(h)d\mu_G(x)$$

Applying the Fubini-Tonelli theorem, Lemma 2.25, and our assumption on modular functions,

$$\begin{aligned} \int_G \int_H f(xh)g(x)d\mu_H(h)d\mu_G(x) &= \int_H \int_G f(xh)g(x)d\mu_G(x)d\mu_H(h) \\ &= \int_H \Delta_G(h^{-1}) \int_G f(x)g(xh^{-1})d\mu_G(x)d\mu_H(h) \\ &= \int_G f(x) \int_H \Delta_H(h^{-1})g(xh^{-1})d\mu_H(h)d\mu_G(x). \end{aligned}$$

Lemma 2.26 ensures that $\int_H \Delta_H(h^{-1})g(xh^{-1})d\mu_H(h) = \int_H g(xh)d\mu_H(h)$. Hence,

$$\int_G \int_H f(xh)g(x)d\mu_H(h)d\mu_G(x) = \int_G f(x)g^H(x)d\mu_G(x).$$

There is $g \in C_c(G)$ such that $g^H = 1$ on $\mathrm{supp}(f)$, so we deduce that $\int_G f(x)d\mu_G(x) = 0$. Hence, the Haar integral is well-defined on $C_c(G)/K$.

We may now define a linear functional $I : C_c(G/H) \to \mathbb{C}$ by

$$I(g) := \int_G \tilde{\Psi}^{-1}(f)d\mu_G.$$

The map I is a positive left invariant linear functional. Therefore, Riesz's representation theorem supplies an invariant outer Radon measure ν on G/H. □

Our final theorem shows how the invariant measure ν in Theorem 2.29 interpolates between the Haar measure of G and that of a closed subgroup H.

Theorem 2.30. *Let G be a t.d.l.c. Polish group and $H \leq G$ a closed subgroup. Supposing that G/H admits a non-zero invariant outer Radon measure, the following hold.*

(1) The non-zero invariant outer Radon measure on G/H is unique up to constant multiples.

(2) Given μ_G and μ_H left Haar measures on G and H respectively, there is a unique choice for ν such that for every $f \in C_c(G)$,

$$\int_G f(x)d\mu_G(x) = \int_{G/H}\int_H f(xh)d\mu_H(h)d\nu(x).$$

This relationship is called the ***quotient integral formula****.*

Proof. This follows from the proof of Theorem 2.29; we leave checking the details as Exercise 2.20 □

Remark 2.31. The quotient integral formula in fact holds for all functions in $L_1(\mu)$. We do not discuss this here as it requires somewhat deeper functional analysis.

Notes

The existence proof given for the Haar measure is due to F. Le Maître. Our approach to the uniqness of Haar measure and the quotient integral formula follows that of A. Deitmar and S. Echterhoff in [6]. The existence and uniqueness results for left or right Haar measure indeed hold for any locally compact group; the interested reader is directed to [6] or [8].

For an excellent introduction to functional analysis and a proof of the Riesz representation theorem, we direct the reader to [7].

2.6 Exercises

Topology and measure theory

Exercise 2.1. Let X and Y be metric spaces with Y a compete metric space. Suppose that $Z \subseteq X$ is dense and $f : Z \to Y$ is such that for any Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X, the image $(f(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in Y. Show there is a unique continuous function $\tilde{f} : X \to Y$ such that the restriction of $\tilde{f}$ to Z is f.

Exercise 2.2. Suppose that X and Y are Polish spaces. A function $f : X \to Y$ a called **Borel measurable** if the preimage of any open set is a Borel set. Show $f^{-1}(A)$ is a Borel set for any Borel set A of Y.

Exercise 2.3. Let X and Y be topological spaces with $f : X \to Y$ a homeomorphism. Show that if X is equipped with an outer Radon measure μ, then $f_*\mu$ is an outer Radon measure on Y.

Exercise 2.4. Verify the function μ defined in Lemma 2.13 is indeed a measure.

Exercise 2.5. Let X be a Polish space and equip $C_c(X)$ with the norm topology. Show $C_c(X)$ is a topological vector space. That is, show the vector space operations are continuous with respect to the norm topology.

Exercise 2.6. Let G be a t.d.l.c. Polish group and $f \in C_c(G)$. Show there is $h \in C_c^+(G)$ such that $h \equiv 1$ on $\operatorname{supp}(f)$. (You may not appeal to Urysohn's lemma.)

Exercise 2.7. For G a topological group, suppose that the sequence $(f_i)_{i\in\mathbb{N}}$ of elements of $C_c(G)$ converges to some $f \in C_c(G)$ in the uniform topology. Show $L_g(f_i) \to L_g(f)$ and $R_g(f_i) \to R_g(f)$ for any $g \in G$.

Exercise 2.8. Let X be a Polish space and Φ be a positive linear functional on $C_c(X)$. Show Φ is continuous.

Haar measure

Exercise 2.9. Let G be a compact Polish group and $H \trianglelefteq_o G$. Show

$$F := \{f : G \to \mathbb{C} \mid f \text{ is continuous and } f(x) = f(xh) \text{ for all } h \in H\}$$

is isomorphic as a vector space to $L := \{f : G/H \to \mathbb{C}\}$.

Exercise 2.10. A measure space (X, μ) is **σ-finite** if X is a countable union of finite measure sets. For a t.d.l.c. Polish group, show any Haar measure is sigma finite. Does the same hold for all t.d.l.c. groups?

Exercise 2.11. Let G be a t.d.l.c. Polish group. Show that if G is abelian or compact, then $\Delta \equiv 1$.

Exercise 2.12. Let G be a t.d.l.c. Polish group. Show that if G admits a Haar measure that is both left and right invariant, then every Haar measure is both left and right invariant.

Exercise 2.13. Complete this exercise without appealing to the uniqueness of the Haar measure. Suppose that μ and ν are left Haar measures on a t.d.l.c. Polish group G and that there is $c \in \mathbb{R}_{>0}$ such that $\int_G f d\mu = c \int_G f d\nu$ for all $f \in C_c(G)$. Show that $\mu = c\nu$.

Exercise 2.14. Let G be a t.d.l.c. Polish group with left Haar measure μ. Show the following are equivalent:

(1) There is $x \in G$ such that $\mu(\{x\}) > 0$.

(2) The set $\{1\}$ has positive measure.

(3) The Haar measure is a multiple of counting measure.

(4) G is a discrete group.

Exercise 2.15. Prove Lemma 2.15.

Exercise 2.16. Let G be a t.d.l.c. Polish group with Haar measure μ. Show $\mu(G) < \infty$ if and only if G is compact.

Exercise 2.17. Let G be a t.d.l.c. Polish group and μ the left Haar measure on G. For $g \in G$ and $f \in L^1(G)$, show the following:

(a) $R_g(f)$ and $L_g(f)$ are elements of $L^1(G)$.

(b) $\int_G R_g(f(x))d\mu(x) = \Delta(g^{-1}) \int_G f(x)d\mu(x)$ where Δ is the modular function.

(c) $\int_G L_g(f(x))d\mu(x) = \int_G f(x)d\mu(x)$. (This shows the left Haar integral is left-invariant as a linear functional on $L^1(G)$.)

Hint: First consider simple functions and then approximate.

Exercise 2.18. Find an elementary proof of Lemma 2.16 for the case of $f \in C_c(G)$.

Exercise 2.19. Let G be a t.d.l.c. Polish group and H be a closed subgroup with left Haar measure μ_H. Define the map $\Psi : C_c(G) \to C_c(G/H)$ by $f(x) \mapsto \int_H f(xh)d\mu_H(h)$.

(a) Show $\mathrm{supp}(\Psi) = \mathrm{supp}(f)H/H$.

(b) Show Ψ is continuous and linear.

Exercise 2.20. Fill in the details for the proof of Theorem 2.30.

Exercise 2.21. Let G be a t.d.l.c. Polish group with left Haar measure μ. Show the modular function Δ only takes rational values. Argue further that $G/\ker(\Delta)$ is a discrete abelian group.

Exercise 2.22 (Weil)**.** Let G be a t.d.l.c. Polish group with left Haar measure μ. Show if $A \subseteq G$ is measurable with $\mu(A) > 0$, then AA^{-1} contains a neighborhood of 1. Use this to prove that if $H \leq G$ is a subgroup with positive measure, then H is open.

HINT: use inner and outer regularity.

Chapter 3

Geometric Structure

This section introduces an important geometric perspective to t.d.l.c. group theory. Our goal is to cast t.d.l.c. groups as geometric objects and then go on to explore some first consequences of this new perspective.

3.1 The Cayley-Abels graph

We being by exploring a type of graph, which we will argue captures the geometric information of compactly generated t.d.l.c. groups. A t.d.l.c. group G is **compactly generated** if there is a compact set $K \subseteq G$ such that $G = \langle K \rangle$.

Definition 3.1. For a t.d.l.c. group G, a locally finite connected graph Γ on which G acts vertex transitively with compact open vertex stabilizers is called a **Cayley-Abels graph** for G.

The existence of a Cayley-Abels graph ensures compact generation.

Proposition 3.2. *If a t.d.l.c. group admits a Cayley-Abels graph, then it is compactly generated.*

Proof. Let G be a t.d.l.c. group with a Cayley-Abels graph Γ. Fix $v \in V\Gamma$ and let $G_{(v)}$ denote the stabilizer of the vertex v in G. The subgroup $G_{(v)}$ is compact and open by the definition of a Cayley-Abels graph.

Since Γ is locally finite and G acts vertex transitively, there are $g_1, \dots, g_n \in G$ such that $\{g_1(v), \cdots, g_n(v)\}$ lists the neighbors of v. Let $F := \langle g_1, \dots, g_n \rangle$.

We now argue by induction on $d_\Gamma(v,w)$ for the claim that there is $\gamma \in F$ such that $\gamma(v) = w$.

The base case, $d_\Gamma(v,w) = 0$, is obvious. Suppose the hypothesis holds up to k and $d_\Gamma(v,w) = k+1$. Let $v, u_1, \cdots, u_k, w$ be a geodesic from v to w. By the induction hypothesis, $u_k = \gamma(v)$ for some $\gamma \in F$. Therefore, $\gamma^{-1}(u_k) = v$, and $\gamma^{-1}(w) = g_i(v)$ for some $1 \leq i \leq n$. We conclude $\gamma g_i(v) = w$ verifying the induction hypothesis.

For all $g \in G$, there is thus $\gamma \in F$ such that $g(v) = \gamma(v)$, so $\gamma^{-1}g \in G_{(v)}$. We conclude that $G = FG_{(v)}$, so $G = \langle g_1, \dots, g_n, G_{(v)} \rangle$ and is compactly generated. $\square$

Much less obviously, the converse of Proposition 3.2 holds. That is to say, every compactly generated t.d.l.c. group admits a Cayley-Abels graph. Our preliminary lemma isolates for a given compactly generated t.d.l.c. group the one ball of such a graph.

Lemma 3.3. *Suppose that G is a compactly generated t.d.l.c. group, X is a compact generating set, and $U \in \mathcal{U}(G)$. Then the following hold:*

(1) There is a finite symmetric set $A \subseteq G$ containing 1 such that $X \subseteq AU$ and $UAU = AU$.

(2) For any finite symmetric set A containing 1 with $X \subseteq AU$ and $UAU = AU$, it is the case that $G = \langle A \rangle U$.

Proof. Since $\{xU \mid x \in X\}$ is an open cover of X, we may find B a finite symmetric set containing 1 such that $X \subseteq BU$. On the other hand, UB is a compact set, so there is a finite symmetric set A containing 1 such that $B \subseteq A \subseteq UBU$ and $UB \subseteq AU$. We conclude that

$$UAU = UBU \subseteq AUU = AU,$$

so $UAU = AU$, verifying (1).

For claim (2), we argue by induction on n that $(UAU)^n = A^nU$ for all $n \geq 1$. The base case is given by our hypotheses. Supposing that $(UAU)^n = A^nU$, we see that

$$(UAU)^{n+1} = (UAU)^nUAU = A^nUUAU = A^nUAU = A^{n+1}U,$$

completing the induction. Since UAU contains X and is symmetric, it now follows that

$$G = \langle UAU \rangle = \bigcup_{n\geq 1}(UAU)^n = \bigcup_{n \geq 1} A^nU = \langle A \rangle U. \qquad \square$$

Remark 3.4. The factorization produced in Lemma 3.3 need not have any algebraic content. The group G need not be an amalgamated free product or semidirect product of $\langle A\rangle$ and U.

Lemma 3.3 suggests a construction of a Cayley-Abels graph. The vertices ought to be left cosets of U, and the set AU, where A is as found in Lemma 3.3, of left cosets of U forms the one ball around coset U. The next theorem makes this intuitive construction precise.

Theorem 3.5 (Abels)**.** *For G a compactly generated t.d.l.c. group and $U \in \mathcal{U}(G)$, there is a Cayley-Abels graph Γ for G such that $V\Gamma = G/U$. In particular, there is $v \in V\Gamma$ such that $G_{(v)} = U$.*

Proof. Applying Lemma 3.3, there is a finite symmetric set A which contains 1 such that $UAU = AU$ and $G = \langle A\rangle U$. We define the graph Γ by $V\Gamma := G/U$ and letting $\beta \in G/U$ be the coset U,

$$E\Gamma := \{\{g\beta, ga\beta\} \mid g \in G \text{ and } a \in A \setminus \{1\}\}\,.$$

We argue Γ satisfies the theorem. It is clear that G acts vertex transitively on Γ with compact open vertex stabilizers; the vertex stabilizers are conjugates of U. It remains to show that Γ is connected and locally finite.

For connectivity, take $g\beta \in V\Gamma$. Lemma 3.3 ensures that $G = \langle A\rangle U$, so we may write $g = a_1 \dots a_n u$ for $a_1, \cdots, a_n$ elements of A and $u \in U$. Thus,

$$\beta, a_1\beta, a_1a_2\beta, \cdots, g\beta$$

is a path in Γ connecting β to $g\beta$. Hence, Γ is connected.

For local finiteness, it suffices to show $B_1(\beta) = \{a\beta \mid a \in A\}$, since G acts on Γ vertex transitively. If $\{k\beta, ka\beta\}$ is an edge in Γ with β as an end point, then either $k\beta = \beta$ or $ka\beta = \beta$. In the former case, $k \in U$, so $kaU \subseteq UAU$. As $UAU = AU$, we conclude that $ka = a'u$ for some $a' \in A$ and $u \in U$. Hence, $ka\beta = a'u\beta = a'\beta$. For the latter case, $k = ua^{-1}$ for some $u \in U$, so $kU \subset UAU = AU$, since A is symmetric. Hence, $k = a'u$ for some $a' \in A$ and $u \in U$, and therefore, $k\beta = a'u\beta = a'\beta$. In either case, the edge $\{k\beta, ka\beta\}$ is of the form $\{\beta, a'\beta\}$ for some $a' \in A$. We conclude that $B_1(\beta) = \{a\beta \mid a \in A\}$ and Γ is locally finite. □

Corollary 3.6 (Abels)**.** *A t.d.l.c. group admits a Cayley-Abels graph if and only if it is compactly generated.*

Proof. The forward implication is given by Proposition 3.2. The reverse is given by Theorem 3.5. $\square$

Remark 3.7. As soon as a compactly generated t.d.l.c. group is non-discrete, the action on a Cayley-Abels graph is *never* free. That is to say, the action always has non-trivial vertex stabilizers. We shall see that these large, but compact stabilizers play an important role in the structure of compactly generated t.d.l.c. groups.

A priori, the graph built in the proof of Theorem 3.5 is just one way to produce a Cayley–Abels graph for a given compactly generated t.d.l.c. group. We close this section by showing that every Cayley–Abels graph is of the same form.

Let us denote the graph built in the proof of Theorem 3.5 by $\Gamma_{A,U}$. That is, U is a compact open subgroup of G and A is a finite symmetric set containing 1 such that AU is a generating set for G and $UAU = AU$. The graph $\Gamma_{A,U}$ is defined by $V\Gamma_{A,U} := G/U$ and letting $\beta \in G/U$ be the coset U,

$$E\Gamma_{A,U} := \{\{g\beta, ga\beta\} \mid g \in G \text{ and } a \in A \setminus \{1\}\}.$$

We call the set AU a **protean generating set** and $\Gamma_{A,U}$ the associated Cayley-Abels graph.

Lemma 3.8. *Suppose that G is a compactly generated t.d.l.c. group and Γ is a Cayley-Abels graph for G. Fix $v \in V\Gamma$, set $U := G_{(v)}$, and let $B \subseteq G$ be finite containing 1 such that $B(v) = B_1(v)$. Setting $A := B \cup B^{-1}$, the following hold:*

(1) $A(v) = B_1(v)$;

(2) $UAU = AU$; and

(3) $G = \langle A \rangle U$.

Proof. For any $b \in B \setminus \{1\}$, the edge $\{b(v), v\}$ is an edge of Γ. Therefore, $\{v, b^{-1}(v)\}$ is an edge in Γ, so $b^{-1}(v) \in B_1(v)$. We conclude that $A(v) = B_1(v)$, verifying (1).

Taking $a \in A$, the vertex $a(v)$ is a member of $B_1(v)$, and as U is the stabilizer of v, ua is also a member of $B_1(v)$. Thus, $ua(v) = a'(v)$ for some $a' \in A$, so $uaU = a'U$. We deduce that $UAU = AU$, verifying (2).

The proof of Proposition 3.2 shows that AU is a generating set for G. Applying Lemma 3.3, we obtain (3). $\square$

For A and U as in Lemma 3.8, we may form the Cayley-Abels graph $\Gamma_{A,U}$. The next lemma shows this is the same graph with which we started.

Lemma 3.9. *Suppose that G is a compactly generated t.d.l.c. group and Γ is a Cayley-Abels graph for G. Fix $v \in V\Gamma$, set $U := G_{(v)}$, let $B \subseteq G$ be finite containing 1 such that $B(v) = B_1(v)$, and put $A := B \cup B^{-1}$. Then there is a G-equivariant graph isomorphism $\psi : \Gamma \to \Gamma_{A,U}$.*

Proof. For each $w \in V\Gamma$, fix $g_w \in G$ such that $g_w(v) = w$. We obtain a bijection $\psi : V\Gamma \to V\Gamma_{A,U}$ by $w \mapsto g_w\beta$. It is clearly the case that $\psi(g(w)) = g\psi(w)$ for all $g \in G$ and $w \in V\Gamma$, so the map ψ is G-equivariant.

Let $\{g_w(v), g_{w'}(v)\} \in E\Gamma$; the case that $\{g_w(v), g_{w'}(v)\} \notin E\Gamma$ is similar. We see that $\{v, g_w^{-1}g_{w'}(v)\} \in E\Gamma$, so $g_w^{-1}g_{w'} = bu$ for some $b \in B$ and $u \in U$. Hence, $\{g_w\beta, g_{w'}\beta\} = \{g_w\beta, g_w b\beta\}$, so $\{g_w\beta, g_{w'}\beta\} \in E\Gamma_{A,U}$. We conclude that ψ is such that $\{w, w'\} \in E\Gamma$ if and only if $\{\psi(w), \psi(w')\} \in \Gamma_{A,U}$, so ψ is a graph isomorphism. $\square$

Our construction of the Cayley-Abels graph is quite similar to the classical construction of a Cayley graph for a finitely generated group. For G a finitely generated group with X a finite symmetric generating set for G, the **Cayley graph** for G with respect to X, denoted by $\mathrm{Cay}(G, X)$, is defined by $V\mathrm{Cay}(G, X) := G$ and

$$E\mathrm{Cay}(G, X) := \{\{g, gx\} \mid g \in G \text{ and } x \in X \setminus \{1\}\}.$$

The graph $\mathrm{Cay}(G, X)$ is locally finite and connected, and G acts vertex transitively on $\mathrm{Cay}(G, X)$. An important difference, however, is that G acts with trivial vertex stabilizers.

Remark 3.10. For G a compactly generated t.d.l.c. group, a Cayley-Abels graph Γ is produced from a factorization $G = \langle A\rangle U$ with A finite and $U \in \mathcal{U}(G)$. The graph Γ is **not** in general the Cayley graph of the finitely generated group $\langle A\rangle$. At the time of this writing, it is indeed unclear what the relationship is between $\mathrm{Cay}(\langle A\rangle, A)$ and Γ.

3.2 Uniqueness

One can produce infinitely many different Cayley-Abels graphs simply by varying the compact open subgroup U in Theorem 3.5. It turns out, however, that the Cayley-Abels graph is unique up to a natural notion of equivalence of metric spaces. We regard the Cayley-Abels graph as a metric space via the natural graph metric discussed in Chapter 1.

Definition 3.11. Let X and Y be metric spaces with metrics d_X and d_Y, respectively. A map $\phi : X \to Y$ is a **quasi-isometry** if there exist real numbers $k \geq 1$ and $c \geq 0$ for which the following two conditions hold:

(a) For all $x, x' \in X$,

$$\frac{1}{k} d_X(x, x') - c \leq d_Y(\phi(x), \phi(x')) \leq k d_X(x, x') + c, \text{ and}$$

(b) for all $y \in Y$, there is $x \in X$ such that $d_Y(y, \phi(x)) \leq c$

When we wish to emphasize the constants k and c, we say that ϕ is a (k, c) quasi-isometry. If there is a quasi-isometry between X and Y, we say they are **quasi-isometric** and write $X \simeq_{qi} Y$.

Quasi-isometries preserve the large scale structure of a metric space while allowing for bounded distortion on small scales. Quasi-isometry is in particular weaker than isometry, since isometries preserve structure on all scales. While not immediately obvious from the definition, the relation $\simeq_{qi}$ is an equivalence relation on metric spaces; see Exercise 3.4.

The uniqueness theorem requires a notion of a quotient for Cayley-Abels graphs. A G**-congruence** σ for a group G acting on a set X is an equivalence relation $\sim_\sigma$ on X such that $x \sim_\sigma y$ if and only if $g(x) \sim_\sigma g(y)$ for all $g \in G$ and $x, y \in X$. The equivalence classes of σ are called the **blocks of imprimitivity** of σ; we call the classes "blocks" for brevity. If $x \in X$, the block containing x is denoted by x^σ.

Suppose $G \curvearrowright \Gamma$ with Γ a connected graph and σ a G-congruence on $V\Gamma$. We define the quotient graph Γ/σ by setting

$$V\Gamma/\sigma := \{v^\sigma \mid v \in V\Gamma\}$$

and $E\Gamma/\sigma$ to be

$$\{\{v^\sigma, w^\sigma\} \mid v^\sigma \neq w^\sigma \text{ and } \exists v' \in v^\sigma \text{ and } w' \in w^\sigma \text{ such that } \{v', w'\} \in E\Gamma\}.$$

The action of G on Γ descends to an action on Γ/σ by $g(v^\sigma) := (g(v))^\sigma$, and this action is by graph automorphisms. Additionally, Γ/σ is connected. See Exercise 3.6.

Remark 3.12. The above quotient operation is a "naive quotient." In Chapter 4, we will introduce a more subtle quotient which is more robust. We use the naive quotient here as it is much less technical and the additional power of the "correct" quotient is not necessary at this point.

Lemma 3.13. *Let G be a t.d.l.c. Polish group and Γ be a Cayley-Abels graph for G. If σ is a G-congruence on $V\Gamma$ with finite blocks, then Γ/σ is a Cayley-Abels graph for G.*

Proof. The group G acts on Γ/σ vertex transitively, and since each block is finite, Γ/σ is locally finite. It remains to show the stabilizer of a vertex is compact and open.

Take $v^\sigma \in V\Gamma/\sigma$ and say that $v^\sigma = \{v_0, \cdots, v_n\}$. As $G_{(v_0,\cdots,v_n)} \leq G_{(v^\sigma)}$, we deduce that $G_{(v^\sigma)}$ is open. Since $G \curvearrowright \Gamma$ transitively, $v^\sigma = \{v, g_1(v), \cdots, g_n(v)\}$ for some $g_1, \cdots, g_n$ in G. Any $g \in G_{(v^\sigma)}$ must send v to an element of v^σ, so $G_{(v^\sigma)}(v) \subseteq \{v, g_1(v), \cdots, g_n(v)\}$. Hence,

$$G_{(v^\sigma)} \subseteq G_{(v)} \cup g_1 G_{(v)} \cup \cdots \cup g_n G_{(v)}.$$

The subgroup $G_{(v^\sigma)}$ is thus also compact. $\square$

We are now prepared to prove the desired uniqueness result.

Theorem 3.14. *The Cayley-Abels graph for a compactly generated t.d.l.c. group is unique up to quasi-isometry.*

Proof. Let Γ and Δ be two Cayley-Abels graphs for a compactly generated t.d.l.c. group G. Fix $r \in V\Gamma$ and $s \in V\Delta$ and set $U := G_{(r)}$ and $V := G_{(s)}$. Lemma 3.8 supplies finite symmetric sets A_U and A_V each containing 1 such that

(i) $A_U(r) = B_1^\Gamma(r)$ and $A_V(s) = B_1^\Delta(s)$;

(ii) $UA_UU = A_UU$ and $VA_VV = VA_V$;

(iii) $G = \langle A_U \rangle U$ and $G = \langle A_V \rangle V$.

We now consider two cases, which will be enough to prove the theorem.

Case (1): $U = V$. Since every $u \in V\Gamma$ is of the form $g(r)$ for some $g \in G$ and $U = V$, we may define $\psi : V\Gamma \to V\Delta$ by $g(r) \mapsto g(s)$. This map is further surjective since each $v \in V\Delta$ is of the form $g(s)$ for some $g \in G$.

For each $a \in A_U$, let $w_a v = a$ be an expression of a in the factorization $G = \langle A_V \rangle V$ where $w_a \in \langle A_V \rangle$ and $v \in V$. Fix $c > \max\{|w_a| \mid a \in A_U\}$ where $|w_a|$ is the word length in the generating set A_V.

Consider $g(r)$ and $h(r)$ with g and h in G and write $g^{-1}h = a_1 \dots a_n u$ where $a_i \in A_U$ and $u \in U = V$. We now see that

$$\begin{aligned} d_\Delta(\psi(g(r)), \psi(h(r))) &= d_\Delta(g(s), h(s)) \\ &= d_\Delta(s, g^{-1}h(s)) \\ &= d_\Delta(s, a_1 \dots a_n(s)). \end{aligned}$$

We may write $a_1 \dots a_n v = w_{a_1} v_1 \dots w_{a_n} v_n$. Since $V(A_V)^n V = (A_V)^n V$ by (ii) above, we may move the v_i terms past the w_{a_j} terms without changing the word length of the w_{a_j} terms; we will in general obtain a new word, however. We thus have $w_{a_1} v_1 \dots w_{a_n} v_n = w'_{a_1} \dots w'_{a_n} v'$ with $w'_{a_i} \in \langle A_V \rangle$, $|w_{a_i}| < c$, and $v' \in V$. Hence,

$$d_\Delta(s, a_1 \dots a_n(s)) = d_\Delta(s, w'_{a_1} \dots w'_{a_n}(s)) \leq cn = c d_\Gamma(g(r), h(r)).$$

On the other hand, for each $b \in A_V$, let $w_b u = b$ be an expression of b in the factorization $G = \langle A_U \rangle U$. Take $c' > \max\{|w_b| \mid b \in A_V\}$. As in the previous paragraph, it follows that

$$d_\Gamma(g(r), h(r)) \leq c' d_\Delta(\psi(g(r)), \psi(h(r))).$$

Putting $k := \max\{c, c'\}$, we conclude that

$$\frac{1}{k} d_\Gamma(g(r), h(r)) \leq d_\Delta(\psi(g(r)), \psi(h(r))) \leq k d_\Gamma(g(r), h(r)),$$

and since ψ is onto, Γ and Δ are quasi-isometric.

Case (2): $U \leq V$. Define an equivalence relation σ on $V\Gamma$ by $g(r) \sim_\sigma h(r)$ if and only if $h^{-1}g \in V$. We see that σ is indeed a G-congruence on $V\Gamma$, and since $|V : U| < \infty$, the blocks are finite. Via Lemma 3.13, Γ/σ is a Cayley-Abels graph for G, and the stabilizer of r^σ is V. Hence, $\Gamma/\sigma \simeq_{qi} \Delta$ by case (1). It thus suffices to show Γ/σ is quasi-isometric to Γ.

Fix $\{h_i(r) \mid i \in \mathbb{N}\}$ equivalence class representatives for $V\Gamma/\sigma$ and let c be strictly greater than the diameter of a (any) block of σ. Define $\phi : \Gamma/\sigma \to \Gamma$ by $\phi(h_i(r)^\sigma) \mapsto h_i(r)$. We outright have

$$d_{\Gamma/\sigma}\big(h_i(r)^\sigma, h_j(r)^\sigma\big) \leq d_\Gamma\big(\phi(h_i(r)^\sigma), \phi(h_j(r)^\sigma)\big).$$

On the other hand, let $v_1^\sigma, \cdots, v_n^\sigma$ be a geodesic from $h_i(r)^\sigma$ to $h_j(r)^\sigma$ in Γ/σ. We now extract a sequence of vertices in Γ from the geodesic $v_1^\sigma, \cdots, v_n^\sigma$; see Figure 3.2. Take $u_i^+ \in v_i^\sigma$ for $1 \leq i < n$ and $u_j^- \in v_j^\sigma$ for $1 < j \leq n$ such that $\{u_k^+, u_{k+1}^-\} \in E\Gamma$ for $1 \leq k < n$. We now have

$$d_\Gamma(h_i(r), h_j(r)) \leq d_\Gamma(h_i(r), u_1^+) + 1 + d_\Gamma(u_2^-, u_2^+) + 1 + \cdots + d_\Gamma(u_n^-, h_j(r)).$$

Since the blocks have diameter strictly less than c, $d_\Gamma(h_i(r), h_j(r)) \leq cn$.

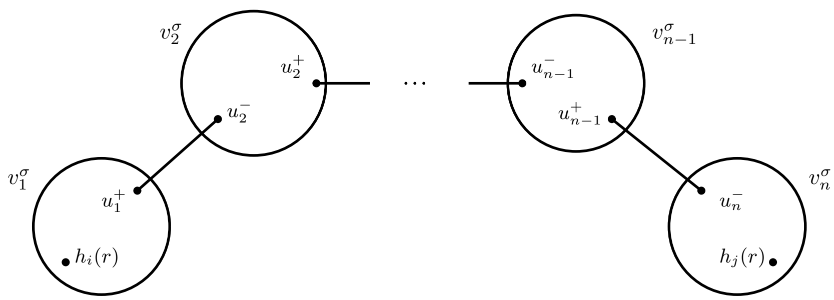

Figure 3.1: The extracted vertices

Hence,

$$\frac{1}{c} d_{\Gamma/\sigma}\left(h_i(r)^\sigma, h_j(r)^\sigma\right) \leq d_\Gamma\left(\phi(h_i(r)^\sigma), \psi(h_i(r)^\sigma)\right) \leq c d_{\Gamma/\sigma}\left(h_i(r)^\sigma, h_j(r)^\sigma\right).$$

Since every $g(r) \in V\Gamma$ lies in some block, there is i such that $d_\Gamma(g(r), h_i(r)) \leq c$. Therefore, ϕ is a quasi-isometry.

The general case is now in hand. Suppose Γ and Δ are Cayley-Abels graphs for G. We may find Φ a Cayley-Abels graph for G on the coset space $G/U \cap V$ by Theorem 3.5. By case two above, Γ and Δ are quasi-isometric to Φ, and therefore, they are quasi-isometric to each other. $\square$

In view of Theorem 3.14, we say t.d.l.c. groups G and H are **quasi-isometric** if some (any) Cayley-Abels graphs for G and H are quasi-isometric. The quasi-isometry type of the Cayley-Abels graph is the geometric structure of a compactly generated t.d.l.c. group. This is what we have in mind when we refer to the "geometric structure" of a t.d.l.c. group.

3.3 Cayley-Abels representations

Given a compactly generated t.d.l.c. group G and a Cayley-Abels graph Γ for G, we obtain a representation - i.e. a homomorphism - $\psi : G \to \mathrm{Aut}(\Gamma)$

via the action of G on Γ. This representation is called a **Cayley-Abels representation** of G. One naturally wishes to understand this representation better, and we here observe some basic properties.

The automorphism group of any locally finite connected graph is a t.d.l.c. group by the results of Chapter 1. We may thus sensibly ask about the continuity of the Cayley-Abels representation. The proof of our first lemma follows easily from the definitions, so we leave it as Exercise 3.9.

Lemma 3.15. *For G a compactly generated t.d.l.c. group with a Cayley-Abels graph Γ, the Cayley-Abels representation $\psi : G \to \mathrm{Aut}(\Gamma)$ is continuous.*

We now consider the image of the Cayley-Abels representation. An important concept here, and generally in the study of locally compact groups, is that of a cocompact subgroup.

Definition 3.16. For G a topological group with H a closed subgroup of G, we say that H is **cocompact** in G if the quotient space G/H equipped with the quotient topology is compact.

It turns out that defining cocompact subgroups via the right coset space $H\backslash G$ is equivalent.

Lemma 3.17. *For G a topological group with H a closed subgroup, the space of left cosets G/H is compact if and only if the space of right cosets $H\backslash G$ is compact.*

Proof. Exercise 3.5 □

Cayley-Abels graphs allow us to easily identify cocompact subgroups.

Lemma 3.18. *Let G be a compactly generated t.d.l.c. group, H be a closed subgroup of G, and Γ be a Cayley-Abels graph for G. Then H is cocompact in G if and only if H has finitely many orbits on $V\Gamma$.*

Proof. Suppose first that H is cocompact in G. The space of right cosets $H\backslash G$ is then compact. Fix $w \in V\Gamma$ and let $U := G_{(w)}$. The collection $\{HgU \mid g \in G\}$ forms an open cover of $H\backslash G$, so there is a finite subcover. We may thus find $X := g_1U \cup \cdots \cup g_nU$ such that $HX = G$. The action of G on $V\Gamma$ is transitive, so $G(w) = V\Gamma$. Hence, $HX(w) = \cup_{i=1}^{n} H(g_i(w)) = V\Gamma$. We conclude that H has finitely many orbits on $V\Gamma$.

Conversely, suppose that H has finitely many orbits on $V\Gamma$. Let $v_1, \cdots, v_n$ be representatives for the orbits of H on $V\Gamma$, fix $w \in V\Gamma$, and fix $g_1, \cdots, g_n$ in G such that $g_i(w) = v_i$ for $1 \leq i \leq n$. For $g \in G$, the vertex $g(w)$ is in some orbit of H, so there is $h \in H$ such that $hg(w) = v_i$ for some $1 \leq i \leq n$. Thus $g_i^{-1}hg \in G_{(w)}$. Setting $F := \{g_1, \cdots, g_n\}$, it follows that $G = HFG_{(w)}$. As $FG_{(w)}$ is compact, the set of right cosets $H\backslash G$ is compact. Hence, H is cocompact in G. $\square$

We now collect our results into a theorem which establishes some basic properties of the Cayley-Abels representation.

Theorem 3.19. *Let G be a compactly generated t.d.l.c. group and Γ be a Cayley-Abels graph for G. Then the induced homomorphism $\psi : G \to \mathrm{Aut}(\Gamma)$ is a continuous and closed map, $\psi(G)$ is cocompact in $\mathrm{Aut}(\Gamma)$, and $\ker(\psi)$ is compact.*

Proof. Lemma 3.15 ensures that ψ is continuous.

Set $H := \mathrm{Aut}(\Gamma)$, fix $A \subseteq G$ closed, and say that $\psi(a_i) \to h \in H$. Fixing $w \in V\Gamma$, there is N such that $\psi(a_i^{-1})\psi(a_j) \in H_{(w)}$ for all $i, j \geq N$, and thus, $a_i^{-1}a_j \in G_{(w)}$ for all $i, j \geq N$. Fix $i \geq N$. As $G_{(w)}$ is compact and $a_i^{-1}a_j \in G_{(w)}$ for all $j \geq N$, there is a convergent subsequence $a_i^{-1}a_{j_k} \to b$. The subsequence a_{j_k} thus converges to some $a \in A$, and since ψ is continuous, $\psi(a) = h$. We conclude that ψ is a closed map.

The image $\psi(G)$ is now a closed subgroup of H that acts vertex transitively on $V\Gamma$. Applying Lemma 3.18, we conclude that $\psi(G)$ is cocompact in H. The final claim is immediate since vertex stabilizers are compact. $\square$

Remark 3.20. Since the automorphism groups of locally finite connected graphs are Polish, Theorem 3.19 ensures that every compactly generated t.d.l.c. group is compact-by-Polish. This observation shows our frequent restriction to Polish groups loses little generality.

Let us make a useful observation about cocompact subgroups. For Γ a connected graph and $c \geq 0$, we say that $A \subseteq V\Gamma$ is c**-dense** in Γ if for every $w \in V\Gamma$ there is $a \in A$ such that $d_\Gamma(w, a) \leq c$.

Proposition 3.21. *For G a compactly generated t.d.l.c. group, if H is a closed and cocompact subgroup of G, then H is compactly generated.*

Proof. Fix Γ a Cayley-Abels graph for G.

In view of Lemma 3.18, the subgroup H has finitely many orbits on $V\Gamma$. Let $v_1, \cdots, v_n$ list representatives of the orbits of H on $V\Gamma$ and let O be the orbit of v_1 under the action of H. Taking $w \in V\Gamma$, there is $h \in H$ such that $h(w) \in \{v_1, \cdots, v_n\}$. Hence, $d_\Gamma(h(w), v_1) \leq m$. We conclude that $d_\Gamma(w, h^{-1}(v_1)) \leq m$, and since $h^{-1}(v_1) \in O$, the orbit O is m-dense in Γ.

We now argue that for any two $r, s \in O$, there is a sequence of vertices $r := u_0, \cdots, u_{n+1} =: s$ in O such that $d_\Gamma(u_i, u_{i+1}) \leq 2m+1$ for $0 \leq i \leq n$. Let $r = w_0, \dots, w_{n+1} = s$ be a geodesic from r to s in Γ. Since O is m-dense, we may find $u_i \in O$ for $1 \leq i \leq n$ such that $d(w_i, u_i) \leq m$. The triangle inequality ensures that

$$d_\Gamma(u_i, u_{i+1}) \leq d(u_i, w_i) + d(w_i, w_{i+1}) + d(w_{i+1}, u_{i+1}) \leq 2m+1$$

for $1 \leq i < n$. Setting $u_0 := r$ and $u_{n+1} := s$, the triangle inequality similarly ensures that $d(u_i, u_{i+1}) \leq m+1$ when $i \in \{0, n\}$. We have thus produced the desired sequence of vertices.

We define a new graph Φ by $V\Phi := O$ and

$$E\Phi := \{\{v, w\} \mid v, w \in O \text{ and } 1 \leq d_\Gamma(v, w) \leq 2m+1\}.$$

By the previous paragraph, Φ is connected. Furthermore, H acts vertex transitively with compact open vertex stabilizers on Φ, and one checks that Φ is locally finite. We conclude that Φ is a Cayley-Abels graph for H, and thus, H is compactly generated by Proposition 3.2. □

Cocompact subgroups are also quasi-isometric to the supergroup. The proof follows from an easy analysis of the proof of Proposition 3.21, so we leave it to the reader.

Proposition 3.22. *Let G be a compactly generated t.d.l.c. group. If H is a closed and cocompact subgroup of G, then H is quasi-isometric to G.*

Proof. Exercise 3.10 □

Notes

The notion of a Cayley-Abels graph first appeared in the work of H. Abels [1] in the early 1970s. The work of Abels, however, takes a somewhat technical approach via compactifications. There were several refinements of Abels' works in the intervening years, which eventually led to the approach given here. The approach given here should likely be attributed to B. Krön and R. Möller [10].

In several works, the Cayley-Abels graph is called the *rough Cayley graph*. The term "Cayley-Abels graphs" seems to be the accepted nomenclature.

3.4 Exercises

Exercise 3.1. Let G be a t.d.l.c. group, Γ be a Cayley-Abels graph for G, and N be a closed normal subgroup of G. Show the orbits of N on $V\Gamma$ form a G-congruence.

Exercise 3.2. Let G be a t.d.l.c. group, Γ be a Cayley-Abels graph for G, and N a closed normal subgroup of G. Show the natural projection $\pi : V\Gamma \to V\Gamma/N$ by $gU \mapsto gUN$ is a graph homomorphism.

Exercise 3.3. Let Γ and Δ be connected graphs and $f : \Gamma \to \Delta$ a graph homomorphism. Show for all $x, y \in V\Gamma$, $d_\Gamma(x, y) \geq d_\Delta(f(x), f(y))$.

Exercise 3.4. Show the relation of quasi-isometry is an equivalence relation on the class of metric spaces.

Exercise 3.5. Prove Lemma 3.17

Exercise 3.6. Let G be a group acting on a connected graph Γ. Show that if σ is a G-congruence on Γ, then Γ/σ is connected and G acts on Γ/σ by graph automorphisms.

Exercise 3.7. Let G be a compactly generated t.d.l.c. group and Γ a be Cayley-Abels graph for G. Fix $o \in V\Gamma$ and define

$$V_n := \{g \in G \mid d_\Gamma(o, go) \leq n\}.$$

For each $n \geq 1$, show $g \in V^n$ if and only if $d_\Gamma(o, go) \leq n$. Show further V_1 is compact and a generating set for G.

Exercise 3.8. Let Γ be a vertex transitive, connected, and locally finite graph. Fix $n \geq 1$ and define the graph Γ_n by $V\Gamma_n := V\Gamma$ and

$$E\Gamma_n := \{\{v, w\} \mid 1 \leq d_\Gamma(v, w) \leq n\}.$$

(a) Show Γ_n is quasi-isometric to Γ.

(b) Show if Γ is also a Cayley-Abels graph for a t.d.l.c. group G, then Γ_n is a Cayley-Abels graph for G.

Exercise 3.9. Show the Cayley-Abels representation is continuous.

Exercise 3.10. Prove Proposition 3.22

Exercise 3.11. Prove Lemma **??**.

Exercise 3.12. Let G be a t.d.l.c. group, Γ a Cayley-Abels graph for G, $N \trianglelefteq G$, and $\pi : \Gamma \to \Gamma/N$ be the usual projection. Show for any path γ in Γ, $\pi(\breve{\gamma})$ equals the reverse of $\pi(\gamma)$.

Exercise 3.13. Suppose G is a compactly generated t.d.l.c. Polish group, $H \trianglelefteq G$ is a closed normal subgroup, and Γ is a Cayley-Abels graph for G. Show the following:

(a) The orbits of H on Γ form a G-congruence, denoted by σ.

(b) Γ/σ is locally finite and $\deg(\Gamma/\sigma) \leq \deg(\Gamma)$.

Exercise 3.14. Suppose G is compactly generated and H is a dense subset of G. Show for all $U \in \mathcal{U}(G)$, there is a finite set $F \subseteq H$ such that $G = \langle F \rangle U$. Conclude that for every dense subgroup H of G and Cayley-Abels graph Γ of G, there is a finitely generated subgroup $K \leq H$ that acts transitively on Γ.

Exercise 3.15. For G a t.d.l.c. group, define $B(G) := \{g \in G \mid \overline{g^G} \text{ is compact}\}$, where g^G is the conjugacy class of g in G.

(a) Show $B(G)$ is a characteristic subgroup of G - i.e. $B(G)$ is preserved by every topological group automorphism of G, so in particular it is normal.

(b) Suppose that G is compactly generated and fix Γ a Cayley-Abels graph for G. Show

$$B(G) = \{g \in G \mid \exists N \; \forall v \in V\Gamma \; d_\Gamma(v, g(v)) \leq N\}.$$

(c) Show that if $g \in B(G)$ is such that $\overline{\langle g \rangle}$ is compact, then $\overline{\langle g^G \rangle}$ is compact.

(d) (Challenge) Exhibit an example showing $B(G)$ need not be closed for non-compactly generated t.d.l.c. groups. (We shall see in Exercise 4.16 that $B(G)$ is closed for compactly generated G.)

Chapter 4

Essentially Chief Series

A basic concept in (finite) group theory is that of a chief factor.

Definition 4.1. A **normal factor** of a (topological) group G is a quotient K/L such that K and L are distinct (closed) normal subgroups of G with $L < K$. We say that K/L is a (topological) **chief factor** of G if there is no (closed) normal subgroup M of G such that $L < M < K$.

In finite group theory, chief factors play an essential role in the classical structure theory.

Fact 4.2. *Every finite group F admits a finite series $\{1\} = F_0 < F_1 < \dots F_n = F$ of normal subgroups of F such that each normal factor F_i/F_{i-1} is a chief factor.*

The series given in the above fact is called a chief series. Such a series additionally enjoys a uniqueness property.

Fact 4.3 (Jordan–Hölder)**.** *The chief factors appearing in a chief series of a finite group are unique up to permutation and isomorphism.*

In this chapter, we will see that compactly generated t.d.l.c. groups admit a close analogue of the chief series which additionally enjoys a uniqueness property.

4.1 Graphs revisited

4.1.1 A new definition

Our results here require a more technical, but more powerful, notion of a graph. This additional complication is necessary for the desired results to ensure the degree of a graph behaves well under quotients. The notion of a graph given here seems to be the metamathematically "correct" notion of a graph, in this author's opinion.

Definition 4.4. A **graph** $\Gamma = (V\Gamma, E\Gamma, o, r)$ consists of a set $V\Gamma$ called the **vertices**, a set $E\Gamma$ called the **edges**, a map $o : E\Gamma \to V\Gamma$ assigning to each edge an **initial vertex**, and a bijection $r : E\Gamma \to E\Gamma$, denoted by $e \mapsto \overline{e}$ and called **edge reversal**, such that $r^2 = \mathrm{id}$.

Given a classical graph, i.e. a graph as defined in Chapter 1, we produce a graph in the sense above by replacing each unordered edge $\{v, w\}$ by the ordered pairs (v, w) and (w, v). The initial vertex map o is defined to be the projection on the first coordinate, and the edge reversal map sends (v, w) to (w, v).

Convention. For the remainder of this chapter, the term "graph" shall refer to Definition 4.4.

Remark 4.5. We may see classical graphs as graphs in our sense here. Our new definition, however, allows for much more exotic graphs. For example, our new definition of a graph allows for graphs with loops and multiple edges between two vertices.

The **terminal vertex** of an edge is defined to be $t(e) := o(\overline{e})$. A **loop** is an edge e such that $o(e) = t(e)$. For e a loop, we allow both $\overline{e} = e$ and $\overline{e} \neq e$ as possibilities. For a vertex $v \in V\Gamma$, we define

$$E(v) := \{e \in E\Gamma \mid o(e) = v\} = o^{-1}(v);$$

the set $E(v)$ is sometimes called the **star** at v. The **degree** of v is $\deg(v) := |E(v)|$, and the graph Γ is **locally finite** if every vertex has finite degree. The **degree** of the graph is defined to be

$$\deg(\Gamma) := \sup_{v \in V\Gamma} \deg(v).$$

The graph is **simple** if the map $E \to V \times V$ defined by $e \mapsto (o(e), t(e))$ is injective and no edge is a loop.

A **path** p is a sequence of edges $e_1, \ldots, e_n$ such that $t(e_i) = o(e_{i+1})$ for each $i < n$. The length of the path p, denoted by $l(p)$, is the number of edges n. We say that p is a path from vertex v to vertex w if $o(e_1) = v$ and $t(e_n) = w$. A least length path between two vertices is called a **geodesic**. We say that a graph is **connected** if there is a path between any two vertices.

Connected graphs are metric spaces under the graph metric: the **graph metric** on a connected graph Γ is

$$d_\Gamma(v, u) := \begin{cases} \min\{l(p) \mid p \text{ is a path connecting } v \text{ to } u\} & \text{if } v \neq u \\ 0 & \text{if } v = u \end{cases}.$$

For $v \in V\Gamma$ and $k \geq 1$, the k**-ball** around v is defined to be $B_k(v) := \{w \in V\Gamma \mid d_\Gamma(v, w) \leq k\}$ and the k**-sphere** is defined to be $S_k(v) := \{w \in V\Gamma \mid d_\Gamma(v, w) = k\}$.

An **isomorphism** $\alpha : \Gamma \to \Delta$ between graphs is a pair (α_V, α_E) where $\alpha_V : V\Gamma \to V\Delta$ and $\alpha_E : E\Gamma \to E\Delta$ are bijections such that $\alpha_V(o(e)) = o(\alpha_E(e))$ and $\overline{\alpha_E(e)} = \alpha_E(\overline{e})$. We say that α_V and α_E respect the origin and edge reversal maps. An automorphism of Γ is an isomorphism $\Gamma \to \Gamma$. The collection of automorphisms, denoted by $\mathrm{Aut}(\Gamma)$, forms a group under the obvious definitions of composition and inversion:

$$(\alpha_V, \alpha_E) \circ (\beta_V, \beta_E) := (\alpha_V \circ \beta_V, \alpha_E \circ \beta_E) \text{ and } (\alpha_V, \alpha_E)^{-1} := (\alpha_V^{-1}, \alpha_E^{-1}).$$

The automorphism group, $\mathrm{Aut}(\Gamma)$ acts faithfully on the disjoint union $V\Gamma \sqcup E\Gamma$. As we allow for multiple edges and loops, it can be the case that the action of $\mathrm{Aut}(\Gamma)$ on $V\Gamma$ is *not faithful*. For simple graphs, the edges are completely determined by the initial and terminal vertices, so the map α_E is completely determined by α_V. In general, however, this need not be the case.

Remark 4.6. In practice, we often suppress that $g \in \mathrm{Aut}(\Gamma)$ is formally an ordered pair. This usually amounts to simply writing $g(o(e)) = o(g(e))$ and $\overline{g(e)} = g(\overline{e})$. The important bit here is that $g \in \mathrm{Aut}(\Gamma)$ acts on both $V\Gamma$ and $E\Gamma$ and these actions respect the origin and reversal maps.

Just as in Chapter 1, we make $\mathrm{Aut}(\Gamma)$ into a topological group. For finite tuples $\overline{a} := (a_1, \ldots, a_n)$ and $\overline{b} := (b_1, \ldots, b_n)$ over $V\Gamma \cup E\Gamma$, define

$$\Sigma_{\overline{a},\overline{b}} := \{g \in \mathrm{Aut}(\Gamma) \mid g(a_i) = b_i \text{ for } 1 \leq i \leq n\}.$$

The collection $\mathcal{B}$ of sets $\Sigma_{\overline{a},\overline{b}}$ as $\overline{a}$ and $\overline{b}$ run over finite sequences of elements from $V\Gamma \cup E\Gamma$ forms a basis $\mathcal{B}$ for a topology on $\mathrm{Aut}(\Gamma)$. The topology generated by $\mathcal{B}$ is called the **pointwise convergence topology**. We further recover Theorem 1.24; the proof of which is the obvious adaptation of the proof given for Theorem 1.24.

Theorem 4.7. *Let Γ be a graph. If Γ is locally finite and connected, then $\mathrm{Aut}(\Gamma)$ is a t.d.l.c. Polish group.*

4.1.2 Quotient graphs

Quotient graphs play a central role in our proof of the existence of chief series. Our more technical definition of a graph makes quotient graphs easier to define and work with; in particular, we will be able to make useful statements about the degree of quotient graphs.

Let G be a group acting on a graph Γ. For $v \in V\Gamma$ and $e \in E\Gamma$, the orbits of v and e under G are denoted by Gv and Ge, respectively. The **quotient graph** induced by the action of G, denoted by Γ/G, is defined as follows: the vertex set $V(\Gamma/G)$ is the set of G-orbits on V and the edge set $E(\Gamma/G)$ is the set of G-orbits on E. The origin map $\tilde{o} : E(\Gamma/G) \to E(\Gamma/G)$ is defined by $\tilde{o}(Ge) := Go(e)$; this is well-defined since graph automorphisms send initial vertices to initial vertices. The reversal $\tilde{r} : E(\Gamma/G) \to E(\Gamma/G)$ is given by $Ge \mapsto G\overline{e}$; this map is also well-defined. We will abuse notation and write o and r for $\tilde{o}$ and $\tilde{r}$.

There is a natural setting in which group actions descend to quotient graphs. This requires an abstract fact from permutation group theory. Recall from Chapter 1 that a G-congruence is a G-equivariant equivalence relation.

Lemma 4.8. *If G is a group acting on a set X and $N \trianglelefteq G$ is a normal subgroup, then the orbits of N on X form a G-congruence on X.*

Proof. The orbit equivalence relation on X induced by N is given by $v \sim w$ if and only if there is $n \in N$ such that $n(v) = w$. Fix $g \in G$ and suppose $v \sim w$. Letting $n \in N$ be such that $n(v) = w$, we see that $gn(v) = g(w)$, so $gng^{-1}g(v) = g(w)$. As N is normal, we conclude that $g(v) \sim g(w)$. The converse is immediate as we can act with g^{-1}. $\square$

Lemma 4.9. *Let G be a group acting on a graph Γ. If N is a normal subgroup of G, then G acts on Γ/N by $g(Nv) = Ng(v)$ and $g(Ne) = Ng(e)$.*

Furthermore, the kernel of this action of G on Γ/N contains N, so the action factors through G/N.

Proof. By Lemma 4.8, it follows that these actions are well-defined. One easily verifies that these actions respect the origin and edge reversal maps, so the action is indeed by graph automorphisms. That N acts trivially on Γ/N is immediate. □

Lemma 4.10. *Let G be a group acting on a graph Γ with N a closed normal subgroup of G and form the quotient graph Γ/N.*

(1) For $Nv \in V(\Gamma/N)$, the degree $\deg(Nv)$ equals the number of orbits of $N_{(v)}$ on $E(v)$.

(2) If $\deg(\Gamma)$ is finite, then $\deg(\Gamma/N) \leq \deg(\Gamma)$, with equality if and only if there exists a vertex $v \in V$ of maximal degree such that $N_{(v)}$ acts trivially on $E(v)$.

(3) For $v \in V$, the vertex stabilizer in G of Nv under the induced action $G \curvearrowright \Gamma/N$ is $NG_{(v)}$.

Proof. For (1), let Ne be an edge of Γ/N such that $o(Ne) = Nv$. There then exists $v' \in Nv$ and $e' \in Ne$ such that $o(e') = v'$. Letting $n \in N$ be such that $n(v') = v$, we have $o(n(e')) = v$, so $n(e') \in E(v)$. All edges of Γ/N starting at Nv are thus represented by edges of Γ starting at v. The set $E(Nv)$ thus equals $\{Ne \mid e \in E(v)\}$. Letting $\sim$ be the orbit equivalence relation of $N_{(v)}$ acting on $E(v)$, the map $\beta : E(v)/\sim \to E(Nv)$ by $[e] \mapsto Ne$ is easily verified to be a well-defined bijection. Hence, $\deg(Nv) = |E(v)/\sim|$.

For (2), claim (1) ensures $\deg(Nv) \leq \deg(v)$, and $\deg(Nv) = \deg(v)$ if and only if $N_{(v)}$ acts trivially on $E(v)$. Since $v \in V\Gamma$ is arbitrary, the conclusions for the degree of Γ/N are clear.

For (3), let H be the vertex stabilizer of Nv in Γ/N. It follows that H is simply the setwise stabilizer of Nv regarded as a subset of $V\Gamma$. In view of Lemma 4.8, the set Nv is a block of imprimitivity for the action of G on $V\Gamma$. We infer that $G_{(v)} \leq H$, so $G_{(v)} = H_{(v)}$. That N is transitive on Nv and $N \leq H$ now imply that $NG_{(v)} = H$. □

Our more technical notion of a graph ensures that Claim (1) of Lemma 4.10 holds. Let us consider an example which illustrates that this claim can fail for classical graphs and that it gives important information about the group acting on a given graph, which can be hidden in the classical setting.

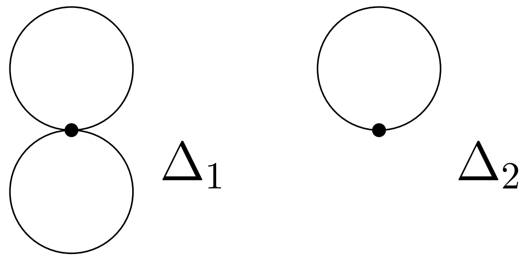

Figure 4.1: The graphs Δ_1 and Δ_2

Example 4.11. Let Γ_c be the classical graph defined by $V\Gamma_c := \mathbb{Z}$ and

$$E\Gamma_c := \{\{i, i+1\} \mid i \in \mathbb{Z}\}.$$

The group of integers $\mathbb{Z}$ and the infinite dihedral group D_∞ act on Γ_c. (The infinite dihedral group is $\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ where $\mathbb{Z}/2\mathbb{Z}$ acts by inversion. The generator of $\mathbb{Z}/2\mathbb{Z}$ acts on the graph Γ_c by reflection over 0.) We now compute the quotient graphs $\Gamma_c/\mathbb{Z}$ and Γ_c/D_∞. The vertex sets of both graphs consist of exactly one vertex, and since loops are not allowed, the edge sets are empty. The quotient graphs are thus the same, and we see no difference between $\mathbb{Z}$ and D_∞ from the perspective of the quotient graph.

Let us next consider the graph $\Gamma = (V\Gamma, E\Gamma, o, r)$ where $V\Gamma := \mathbb{Z}$,

$$E\Gamma := \{(i, j) \mid i, j \in \mathbb{Z} \text{ and } |i - j| = 1\},$$

the origin map is the projection onto the first coordinate, and edge reversal sends (i, j) to (j, i).

We compute the quotient graphs $\Delta_1 := \Gamma/\mathbb{Z}$ and $\Delta_2 := \Gamma/D_\infty$. The vertex sets of both graphs consist of exactly one vertex since $\mathbb{Z}$ and D_∞ both act vertex transitively. The edge set $E\Delta_1$ consists of two edges $e := \mathbb{Z}(0, 1)$ and $f := \mathbb{Z}(1, 0)$ such that $\overline{e} = f$. In particular, $\deg(\Delta_1) = 2$. On the other hand, $E\Delta_2$ is a singleton, since D_∞ acts edge transitively, so $\deg(\Delta_2) = 1$. See Figure 4.1. The quotient graphs, under our more technical notion of a graph, now detect a difference between $\mathbb{Z}$ and D_∞. In view of Claim (1) of Lemma 4.10, the difference detected is exactly that D_∞ has non-trivial vertex stabilizers while $\mathbb{Z}$ does not.

Lemma 4.10 shows that the degree of the quotient graph Γ/N can either become smaller or stay the same. It will be important to gain a deeper insight into the case in which the degree does not decrease under taking a quotient.

Definition 4.12. Given a group G acting on a graph Γ, we say that G acts **freely modulo kernel** on Γ if the vertex stabilizer $G_{(v)}$ acts trivially on both the vertices and the edges of Γ for all $v \in V\Gamma$.

Proposition 4.13. *Let G be a group, N be a normal subgroup of G, and Γ be a connected graph of finite degree on which G acts vertex-transitively. Then, the following are equivalent:*

(1) $\deg(\Gamma/N) = \deg(\Gamma)$.

(2) For some $v \in V\Gamma$, $N_{(v)}$ acts trivially on $E(v)$.

(3) For every $v \in V\Gamma$, $N_{(v)}$ acts trivially on $E(v)$.

(4) N acts freely modulo kernel on Γ.

Proof. (1) $\Rightarrow$ (2). Suppose that $\deg(\Gamma/N) = \deg(\Gamma)$ and fix $v \in V$. Every vertex of Γ/N and Γ has the same degree, since G acts vertex transitively on Γ. Our assumption that $\deg(\Gamma/N) = \deg(\Gamma)$ thus ensures that $|E(Nv)| = |E(v)|$. In view of Lemma 4.10, we conclude that $N_{(v)}$ acts trivially on $E(v)$.

(2) $\Rightarrow$ (3). Suppose that $N_{(v)}$ acts trivially on $E(v)$ and fix $w \in V\Gamma$. Since G acts vertex transitively, we may find $g \in G$ such that $g(v) = w$, and one verifies that $gN_{(v)}g^{-1} = N_{(w)}$, using that N is normal in G. For $e \in E(w)$ and $gng^{-1} \in gN_{(v)}g^{-1} = N_{(w)}$, we see that $g^{-1}(e) \in E(v)$, so $gng^{-1}(e) = e$. Hence, $N_{(w)}$ acts trivially on $E(w)$.

(3) $\Rightarrow$ (4). Say that $N_{(w)}$ acts trivially on $E(w)$ for every $w \in V\Gamma$. Fixing $v \in V\Gamma$, each $g \in N_{(v)}$ fixes $t(e)$, so g fixes the one sphere around v. We conclude that $N_{(v)} \leq N_{(w)}$ for each $w \in S_1(v)$. Inducting on the distance $d_\Gamma(v, w)$, we deduce that $N_{(v)} \leq N_{(w)}$ for every $w \in V\Gamma$; that Γ is connected gives us the metric d_Γ. The vertex stabilizer $N_{(v)}$ thus acts trivially on $V\Gamma$. For any $e \in E\Gamma$, the vertex stabilizer $N_{(o(e))}$ fixes e, and as $N_{(v)} \leq N_{(o(e))}$, we conclude that $N_{(v)}$ fixes e. Hence, $N_{(v)}$ acts trivially on $E\Gamma$, so N acts freely modulo kernel on Γ.

(4) $\Rightarrow$ (1). Say that N acts freely modulo kernel on Γ. Fixing $v \in V\Gamma$, the vertex stabilizer $N_{(v)}$ acts trivially on $E\Gamma$, so a fortiori, $N_{(v)}$ acts trivially on $E(v)$. Lemma 4.10 ensures that $\deg(Nv) = \deg(v)$. Since G acts vertex transitively, we deduce that $\deg(\Gamma/N) = \deg(\Gamma)$. $\square$

4.2 Chain conditions

Given a group G acting on a graph Γ and $N \trianglelefteq G$, Lemma 4.9 allows us to produce from Γ a graph on which G/N acts. For G a compactly generated t.d.l.c. group, Γ a Cayley–Abels graph for G and $N \trianglelefteq G$ closed, one hopes that Γ/N is a Cayley–Abels graph for G/N. This is indeed the case.

Proposition 4.14. *Let G be a compactly generated t.d.l.c. group with N a closed normal subgroup of G. If Γ a Cayley–Abels graph for G, then Γ/N is a Cayley–Abels graph for G/N.*

Proof. As paths in Γ induce paths in Γ/N, the graph Γ/N is connected, and G clearly acts vertex-transitively on Γ/N. Lemma 4.10 ensures that $\deg(\Gamma/N)$ is also finite, so Γ/N is connected and locally finite. Applying Lemma 4.10 a second time, we see that the vertex stabilizer of Nv in G/N is $G_{(v)}N/N$ which is compact. The graph Γ/N is therefore a Cayley–Abels graph for G/N. $\square$

A **filtering family** $\mathcal{F}$ in a partial order $(\mathcal{P}, \leq)$ is a subset of $\mathcal{P}$ such that for any $a, b \in \mathcal{F}$ there is $c \in \mathcal{F}$ for which $c \leq a$ and $c \leq b$. A **directed family** $\mathcal{D}$ is a subset of $\mathcal{P}$ such that for any $a, b \in \mathcal{D}$ there is $c \in \mathcal{D}$ for which $a \leq c$ and $b \leq c$.

We here consider filtering families and directed families of closed normal subgroups of a compactly generated t.d.l.c. group. For filtering or directed families of subgroups, the partial order is always taken to be set inclusion.

Lemma 4.15. *Let G be a compactly generated t.d.l.c. group and Γ be a Cayley–Abels graph for G.*

(1) For $\mathcal{F}$ a filtering family of closed normal subgroups of G and $M := \bigcap \mathcal{F}$, there exists $N \in \mathcal{F}$ such that $\deg(\Gamma/N) = \deg(\Gamma/M)$.

(2) For $\mathcal{D}$ a directed family of closed normal subgroups of G and $M := \overline{\langle \mathcal{D} \rangle}$, there exists $N \in \mathcal{D}$ such that $\deg(\Gamma/N) = \deg(\Gamma/M)$.

Proof. Fix $v \in V\Gamma$ and set $X := E(v)$. The action of the stabilizer $G_{(v)}$ on X induces a homomorphism $\alpha : G_{(v)} \to \mathrm{Sym}(X)$. This map is additionally continuous when $\mathrm{Sym}(X)$ is equipped with the discrete topology; see Exercise 4.4. For N a closed normal subgroup of G, the image $\alpha(N_{(v)}) =: \alpha_N$ is the subgroup of $\mathrm{Sym}(X)$ induced by $N_{(v)}$ acting on X. In view of Lemma 4.10, if $\alpha_N = \alpha_M$ for $M \trianglelefteq G$, then $\deg(\Gamma/N) = \deg(\Gamma/M)$. For a filtering or directed family $\mathcal{N} \subseteq \mathcal{N}(G)$, the family $\alpha(\mathcal{N}) := \{\alpha_N \mid N \in \mathcal{N}\}$ is a filtering or directed family of subgroups of $\mathrm{Sym}(X)$. That $\mathrm{Sym}(X)$ is a finite group ensures that $\alpha(\mathcal{N})$ is a finite family, so $\alpha(\mathcal{N})$ admits a minimum or maximum, according to whether $\mathcal{N}$ is filtering or directed.

Claim (2) is now immediate. The directed family $\alpha(\mathcal{D})$ admits a maximal element α_N. Recalling that $\alpha : G_{(v)} \to \mathrm{Sym}(X)$ is continuous, $\alpha^{-1}(\alpha_N) \cap M$ is closed, and it contains $\langle \mathcal{D} \rangle_{(v)}$ which is dense in $M_{(v)}$. Hence, $\alpha^{-1}(\alpha_N) = M_{(v)}$, and $\alpha_M = \alpha_N$. We conclude that $\deg(\Gamma/N) = \deg(\Gamma/M)$.

For claim (1), an additional compactness argument is required. If G acts freely modulo kernel on Γ, then members N and M of $\mathcal{F}$ also act freely modulo kernel. The desired result then follows since $\deg(\Gamma/N) = \deg(\Gamma) = \deg(\Gamma/M)$. Let us assume that G does not act freely modulo kernel, so $G_{(v)}$ acts non-trivially on $E(v)$ for any $v \in V\Gamma$, via Proposition 4.13.

Take $\alpha(N) \in \alpha(\mathcal{F})$ to be the minimum. Given $r \in \alpha(N)$, let Y be the set of elements of $G_{(v)}$ that *do not* induce the permutation r on X. If $r \neq 1$, then plainly $Y \neq G_{(v)}$. If $r = 1$, then $Y \neq G_{(v)}$ since $G_{(v)}$ acts non-trivially on Γ. The set Y is a proper open subset of $G_{(v)}$, and thus $G_{(v)} \setminus Y$ is a non-empty compact set.

Letting $\mathcal{K}$ be a finite subset of $\mathcal{F}$, the group $K := \bigcap_{F \in \mathcal{K}} F$ contains some element N of $\mathcal{F}$, so $\alpha(K) \geq \alpha(N)$. In particular, $K_{(v)} \not\subseteq Y$. The intersection

$$\bigcap_{F \in \mathcal{K}} (F_{(v)} \cap (G_{(v)} \setminus Y))$$

is therefore non-empty, by compactness. Hence,

$$M_{(v)} \cap (G_{(v)} \setminus Y) = \bigcap_{F \in \mathcal{F}} (F_{(v)} \cap (G_{(v)} \setminus Y)) \neq \emptyset;$$

that is, some element of $M_{(v)}$ induces the permutation r on X. Since $r \in \alpha(N)$ is arbitrary, we conclude that $\alpha(M) = \alpha(N)$, and so $\deg(\Gamma/N) = \deg(\Gamma/M)$. $\square$

In view of Proposition 4.13, the conclusion of claim (1) in Lemma 4.15 implies that the factor N/M is discrete from the point of view of the Cayley–Abels graph.

Lemma 4.16. *Let G be a compactly generated t.d.l.c. group with N a closed normal subgroup of G. If there is a Cayley–Abels graph Γ for G such that*

$\deg(\Gamma/N) = \deg(\Gamma)$, *then there exists a compact normal subgroup L of G acting trivially on Γ such that L is an open subgroup of N.*

Proof. In view of Proposition 4.13, N acts freely modulo kernel on Γ. For U the pointwise stabilizer of the star $E(v)$ for some vertex v, the subgroup U is a compact open subgroup of G, and its core K is the kernel of the action of G on Γ. Since N acts freely modulo kernel, we deduce that $N \cap U \leq K$. The group $L := K \cap N$ now satisfies the lemma. □

Combining Lemmas 4.15 and 4.16, we obtain a result that applies to compactly generated t.d.l.c. groups without dependence on a choice of Cayley–Abels graph.

Theorem 4.17. *Let G be a compactly generated t.d.l.c. group.*

(1) If $\mathcal{F}$ is a filtering family of closed normal subgroups of G, then there exists $N \in \mathcal{F}$ and a closed normal subgroup K of G such that $\bigcap \mathcal{F} \leq K \leq N$, $K/\bigcap \mathcal{F}$ is compact, and N/K is discrete.

(2) If $\mathcal{D}$ is a directed family of closed normal subgroups of G, then there exists $N \in \mathcal{D}$ and a closed normal subgroup K of G such that $N \leq K \leq \overline{\langle \mathcal{D} \rangle}$, K/N is compact, and $\overline{\langle \mathcal{D} \rangle}/K$ is discrete.

Proof. Fix Γ a Cayley–Abels graph for G.

For (1), suppose that $\mathcal{F}$ is a filtering family of closed normal subgroups of G and put $M := \bigcap \mathcal{F}$. Via Lemma 4.15, there is $N \in \mathcal{F}$ such that $\deg(\Gamma/N) = \deg(\Gamma/M)$. The graph Γ/M is a Cayley–Abels graph for G/M by Proposition 4.14. Furthermore, $\deg((\Gamma/M)/(N/M)) = \deg(\Gamma/N) = \deg(\Gamma/M)$; see Exercise 4.5. We may now apply Lemma 4.16 to $N/M \trianglelefteq G/M$. There is thus a closed $K \trianglelefteq G$ such that $M \leq K \leq N$, K/M is compact, and N/K is discrete.

For (2), suppose that $\mathcal{D}$ is a directed family of closed normal subgroups of G and put $L := \overline{\langle \mathcal{D} \rangle}$. Via Lemma 4.15, there is $N \in \mathcal{D}$ such that $\deg(\Gamma/L) = \deg(\Gamma/N)$. The argument now follows as in Claim (1). The quotient graph Γ/N is a Cayley-Abels graph for the quotient group G/N. Furthermore, $\deg((\Gamma/N)/(L/N)) = \deg(\Gamma/L) = \deg(\Gamma/N)$. Applying Lemma 4.16, we obtain a closed $K \trianglelefteq G$ such that $N \leq K \leq L$, K/N is compact, and L/K is discrete. □

4.3 Existence of essentially chief series

Definition 4.18. An **essentially chief series** for a topological group G is a finite series

$$\{1\} = G_0 \leq G_1 \leq \cdots \leq G_n = G$$

of closed normal subgroups such that each normal factor G_{i+1}/G_i is either compact, discrete, or a chief factor of G.

We will see that any compactly generated t.d.l.c. group admits an essentially chief series. In fact, any series of closed normal subgroups can be refined to be an essentially chief series.

Lemma 4.19. *Let G be a compactly generated t.d.l.c. group, H and L be closed normal subgroups of G with $H \leq L$, and Γ be a Cayley–Abels graph for G. Then there exists a series*

$$H =: C_0 \leq K_0 \leq D_0 \leq \cdots \leq C_n \leq K_n \leq D_n := L$$

of closed normal subgroups of G with $n \leq \deg(\Gamma/H) - \deg(\Gamma/L)$ such that

(1) for $0 \leq l \leq n$, K_l/C_l is compact, and D_l/K_l is discrete; and

(2) for $1 \leq l \leq n$, C_l/D_{l-1} is a chief factor of G.

Proof. Set $k := \deg(\Gamma/H)$ and $m := \deg(\Gamma/L)$. By recursion on i, we build a series of closed normal subgroups of G

$$H =: C_0 \leq K_0 \leq D_0 \leq \cdots \leq C_i \leq K_i \leq D_i \leq L$$

such that claims (1) and (2) hold for $0 \leq l \leq i$ and $1 \leq l \leq i$, respectively, and that there is $i \leq j \leq k - m$ for which D_i is maximal among normal subgroups of G such that $\deg(\Gamma/D_i) = k - j$ and $D_i \leq L$.

For $i = 0$, let $\mathcal{L}$ be the collection of closed normal subgroups R of G such that $H \leq R \leq L$ and $\deg(\Gamma/D_0) = k$. Via Lemma 4.15, chains in $\mathcal{L}$ admit upper bounds, so Zorn's lemma supplies D_0 a maximal element of $\mathcal{L}$. The graph Γ/H is a Cayley–Abels graph for G/H with degree k, and

$$\deg((\Gamma/H)/(D_0/H)) = \deg(\Gamma/D_0) = k.$$

Applying Lemma 4.16, we obtain a closed $K_0 \trianglelefteq G$ such that $H \leq K_0 \leq D_0$ with K_0/H compact, open, and normal in D_0/H. The groups $C_0 = H$, K_0, and D_0 satisfy the requirements of our recursive construction when $i = 0$ with $j = 0$.

Suppose we have built our sequence up to i. By construction, there is $i \leq j \leq k - m$ such that D_i is maximal with $\deg(\Gamma/D_i) = k - j$ and $D_i \leq L$. If $j = k - m$, then the maximality of D_i implies $D_i = L$, and we stop. Else, let $j' > j$ be least such that there is $M \trianglelefteq G$ with $\deg(\Gamma/M) = k - j'$ and $D_i \leq M \leq L$. Zorn's lemma in conjunction with Lemma 4.15 supply $C_{i+1} \trianglelefteq G$ minimal such that $\deg(\Gamma/C_{i+1}) = k - j'$ and $D_i < C_{i+1} \leq L$.

Consider a closed $N \trianglelefteq G$ with $D_i \leq N < C_{i+1}$. We have that

$$\deg(\Gamma/N) = \deg((\Gamma/D_i)/(N/D_i)),$$

so $\deg(\Gamma/N) \leq \deg(\Gamma/D_i) = k - j$, by Lemma 4.10. On the other hand,

$$\deg(\Gamma/C_{i+1}) = \deg((\Gamma/N)/(C_{i+1}/N)),$$

so $k - j' = \deg(\Gamma/C_{i+1}) \leq \deg(\Gamma/N)$, by Lemma 4.10. Hence, $k - j' \leq \deg(\Gamma/N) \leq k - j$.

The minimality of C_{i+1} implies that $k - j' < \deg(\Gamma/N)$. On the other hand, $j' > j$ is least such that there is $M \trianglelefteq G$ with $\deg(\Gamma/M) = k - j'$ and $D_i \leq M \leq L$, so $\deg(\Gamma/N) = k - j$. In view of the maximality of D_i, we deduce that $D_i = N$. The factor C_{i+1}/D_i is thus a chief factor of G.

Applying again Lemma 4.15, there is a closed $D_{i+1} \trianglelefteq G$ maximal such that

$$\deg(\Gamma/D_{i+1}) = k - j'$$

and $C_{i+1} \leq D_{i+1} \leq L$. Lemma 4.16 supplies a closed $K_{i+1} \trianglelefteq G$ such that $C_{i+1} \leq K_{i+1} \leq D_{i+1}$ with K_{i+1}/C_{i+1} compact and open in D_{i+1}/C_{i+1}. This completes the recursive construction.

Our recursive construction halts at some $n \leq k - m$. At this stage, $D_n = L$, verifying the theorem. $\square$

Lemma 4.19 allows us to refine a normal series factor by factor to produce an essentially chief series. We can further bounded the length of this series in terms of a group invariant.

Definition 4.20. If G is a compactly generated locally compact group, the **degree** $\deg(G)$ of G is the smallest degree of a Cayley–Abels graph for G.

Theorem 4.21 (Reid–W.)**.** *Suppose that G is a compactly generated t.d.l.c. group. If $(G_i)_{i=1}^{m-1}$ is a finite ascending sequence of closed normal subgroups of G, then there exists an essentially chief series for G*

$$\{1\} = K_0 \leq K_1 \leq \cdots \leq K_l = G,$$

such that $\{G_1, \ldots, G_{m-1}\}$ is a subset of $\{K_0, \ldots, K_l\}$ and $l \leq 2m + 3\deg(G)$. Additionally, at most $\deg(G)$ of the factors K_{i+1}/K_i are neither compact nor discrete.

Proof. Let us extend the series $(G_i)_{i=1}^{m-1}$ by $G_0 := \{1\}$ and $G_m := G$ to obtain the series

$$\{1\} =: G_0 \leq G_1 \leq \cdots \leq G_{m-1} \leq G_m := G.$$

Fix Γ a Cayley-Abels graph for G such that $\deg(G) = \deg(\Gamma)$. For each $j \in \{0, \ldots, m-1\}$, we apply Lemma 4.19 to $L := G_{j+1}$ and $H := G_j$. This produces the essentially chief series $\{1\} = K_0 \leq K_1 \leq \cdots \leq K_l = G$ for G. We now argue that l has the claimed bound.

For each $0 \leq j \leq m$, put $k_j := \deg(\Gamma/G_j)$. In view of Lemma 4.19, the number of new normal subgroups added strictly between G_j and G_{j+1} is at most $3(k_j - k_{j+1}) + 1$, and at most $k_j - k_{j+1}$ of the factors are neither compact nor discrete. The total number of terms in the essentially chief series not including G_m is thus at most

$$\begin{array}{rcl} \sum_{j=0}^{m-1}(3(k_j - k_{j+1}) + 2) & = & 2m + 3(\deg(\Gamma) - \deg(\Gamma/G)) \\ & \leq & 2m + 3\deg(G), \end{array}$$

and the total number of non-compact, non-discrete factors is at most

$$\sum_{j=0}^{m-1}(k_j - k_{j+1}) \leq \deg(G).$$

It now follows that $l \leq 2m + 3\deg(G)$.

□

Corollary 4.22 (Existence of essentially chief series)**.** *Every compactly generated t.d.l.c. group admits an essentially chief series.*

4.4 Uniqueness of essentially chief series

The uniqueness result for essential chief series takes much more work than the existence theorem, and it is, although not obviously, one of the deepest results so far. The uniqueness property will allow us in Chapter **??** to make striking general statements about normal subgroups of t.d.l.c. Polish groups and in particular uncover the structure of chief factors.

Isomorphism is too restrictive of an equivalence for chief factors in Polish groups. The problem with isomorphism arises from a subtlety in the second isomorphism theorem. For G a Polish group and K and L closed normal subgroups of G, the second isomorphism theorem states that $KL/L \simeq K/K \cap L$ as abstract groups. This statement *does not* hold in a topological sense in the setting of Polish or locally compact groups. The internal product KL is not in general closed, so KL/L fails to be a Polish or locally compact group. We develop a weaker notion of equivalence called association. The relation of association "fixes" the second isomorphism theorem for Polish or locally compact groups by relating $K/K \cap L$ to $\overline{KL}/L$, instead of relating $K/K \cap L$ to KL/L.

For G a group and K/L a normal factor of G, the **centralizer** of K/L in G is defined to be

$$\mathrm{C}_G(K/L) := \{g \in G \mid \forall k \in K \ [g, k] \in L\}$$

where $[g, k]$ is the commutator $gkg^{-1}k^{-1}$. Given a subgroup H of G, we put $\mathrm{C}_H(K/L) := \mathrm{C}_G(K/L) \cap H$.

Definition 4.23. For a topological group G, closed normal factors K_1/L_1 and K_2/L_2 are **associated** if $\mathrm{C}_G(K_1/L_1) = \mathrm{C}_G(K_2/L_2)$.

The association relation is clearly an equivalence relation on normal factors. There is furthermore a key refinement theorem, from which we deduce our uniqueness result. The proof of this theorem is rather technical and requires several new notions, so we delay the proof until the next section.

Theorem 4.24. *Let G be a t.d.l.c. Polish group and K/L be a non-abelian chief factor of G. If*

$$\{1\} = G_0 \leq G_1 \leq \cdots \leq G_n = G$$

is a series of closed normal subgroups in G, then there is exactly one $i \in \{0, \ldots, n-1\}$ for which there exist closed normal subgroups $G_i \leq B \leq A \leq$

G_{i+1} of G such that A/B is a non-abelian chief factor associated to K/L. Specifically, this occurs for the least $i \in \{0, \dots, n-1\}$ such that $G_{i+1} \not\leq \mathrm{C}_G(K/L)$.

Definition 4.25. For G a Polish group and K/L a chief factor of G, we say that K/L is **negligible** if K/L is either abelian or associated to a compact or discrete chief factor.

Negligible chief factors look compact or discrete from the point of view of association, and in our uniqueness theorem, we must ignore these factors. We shall see later that negligible chief factors are either close to compact or close to discrete.

In contrast to the results about existence of chief series, we need not assume that G is compactly generated for our uniqueness theorem, but we do need to assume the group is Polish.

Theorem 4.26 (Reid–W.)**.** *Suppose that G is an t.d.l.c. Polish group and that G has two essentially chief series $(A_i)_{i=0}^m$ and $(B_j)_{j=0}^n$. Define*

$$I := \{i \in \{1, \dots, m\} \mid A_i/A_{i-1} \textit{ is a non-negligible chief factor of } G\}; \textit{and}$$
$$J := \{j \in \{1, \dots, n\} \mid B_j/B_{j-1} \textit{ is a non-negligible chief factor of } G\}.$$

Then there is a bijection $f : I \to J$ where $f(i)$ is the unique element $j \in J$ such that A_i/A_{i-1} is associated to B_j/B_{j-1}.

Proof. Theorem 4.24 provides a function $f : I \to \{1, \dots, n\}$ where $f(i)$ is the unique element of $\{1, \dots, n\}$ such that A_i/A_{i-1} is associated to a non-abelian chief factor C/D such that $B_{f(i)} \leq D \leq C \leq B_{f(i)-1}$.

If $B_{f(i)}/B_{f(i)-1}$ is compact, discrete, or abelian, $C/B_{f(i)}$ is compact, discrete or abelian, as each of these classes of groups is stable under taking closed subgroups. Since these classes are also stable under quotients, C/D is either compact, discrete, or abelian. The chief factor C/D is non-abelian, so it must be the case that $B_{f(i)}/B_{f(i)-1}$ is non-abelian. On the other hand, A_i/A_{i-1} is associated to C/D, hence, C/D is neither compact nor discrete, since A_i/A_{i-1} is non negligible. We thus deduce that $B_{f(i)}/B_{f(i)-1}$ is chief, so A_i/A_{i-1} is associated to $B_{f(i)}/B_{f(i)-1}$. Since association is an equivalence relation, we conclude that $B_{f(i)}/B_{f(i)-1}$ is non-negligible, and therefore, $f(i) \in J$.

We thus have a well-defined function $f : I \to J$. The same argument with the roles of the series reversed produces a function $f' : J \to I$ such that

B_j/B_{j-1} is associated to $A_{f'(j)}/A_{f'(j)-1}$. Since each factor of the first series is associated to at most one factor of the second by Theorem 4.24, we conclude that f' is the inverse of f, hence f is a bijection. □

Corollary 4.27 (Uniqueness of essentially chief series)**.** *The non-negligible chief factors appearing in an essentially chief series of a compactly generated t.d.l.c. Polish group are unique up to permutation and association.*

4.5 The refinement theorem

4.5.1 Normal compressions

Definition 4.28. Let G and H be topological groups. A continuous homomorphism $\psi : G \to H$ is a **normal compression** if it is injective with a dense and normal image. When the choice of ψ is unimportant, we say that H is a normal compression of G.

Normal compressions arise naturally in the study of normal subgroups of topological groups. Say that G is a topological group with K and L closed normal subgroups of G. The map $\psi : K/K\cap L \to \overline{KL}/L$ by $k(K\cap L) \mapsto kL$ is a continuous homomorphism with image KL/L. Hence, ψ is a normal compression, and it is not onto as soon as KL is not closed in G.

Lemma 4.29. *Let G and H be t.d.l.c. Polish groups with $\psi : G \to H$ a normal compression. For any $h \in H$, the map $\phi_h : G \to G$ defined by*

$$\phi_h(g) := \psi^{-1}(h\psi(g)h^{-1})$$

is a topological group automorphism of G.

Proof. We leave it to the reader to verify that ϕ_h is an automorphism of G as an abstract group; see Exercise 4.9. To show that ϕ_h is a topological group automorphism, it suffices to argue that ϕ_h is continuous at 1.

Fixing $U \subseteq G$ a compact open subgroup, we see

$$\begin{aligned} \phi_h^{-1}(U) &= \{g \in G \mid \psi^{-1}(h\psi(g)h^{-1}) \in U\} \\ &= \psi^{-1}(h^{-1}\psi(U)h). \end{aligned}$$

Since ψ is continuous, $\psi(U)$ is compact and so closed. Thus, $W := \psi^{-1}(h^{-1}\psi(U)h)$ is a closed set.

The set W is indeed a closed subgroup of G. Furthermore, U has countable index in G, so W also has countable index in G. Write $G = \bigcup_{i\in\mathbb{N}} g_iW$. The Baire category theorem, Fact 1.10, implies that g_iW is non-meagre for some i. As multiplication by g_i is a homeomorphism of G, we infer that W is non-meagre. The subgroup W is thus somewhere dense, so W has non-empty interior, as it is closed. We conclude that W is open and that ϕ_h is continuous. $\square$

In view of Lemma 4.29, there is a canonical action of H on G.

Definition 4.30. Suppose that G and H are t.d.l.c. Polish groups and $\psi : G \to H$ is a normal compression. We call the action of H on G given by $h.g := \phi_h(g)$ the **ψ-equivariant action** of G on H. When clear from context, we suppress "ψ."

The name ψ-equivariant action is motivated by the following lemma.

Lemma 4.31. *Suppose that G and H are t.d.l.c. Polish groups and $\psi : G \to H$ is a normal compression. Letting H act on G by the ψ-equivariant action and H act on itself by conjugation, the map $\psi : G \to H$ is H-equivariant. That is to say, $\psi(h.g) = h\psi(g)h^{-1}$ for all $h \in H$ and $g \in G$.*

Proof. Exercise 4.12 $\square$

We now argue that the ψ-equivariant action is continuous. This requires a technical lemma.

Lemma 4.32. *Let G and H be t.d.l.c. Polish groups with $\psi : G \to H$ a normal compression. For $U \in \mathcal{U}(G)$ and $g \in G$, $N_H(\psi(gU))$ is open in H.*

Proof. We first argue that $N_H(\psi(U))$ is open. The group G is second countable, so G has countably many compact open subgroups. Lemma 4.29 ensures that $h.U$, where the action is the ψ-equivariant action, is also a compact open subgroup for any $h \in H$. It now follows that $\mathrm{Stab_H(U)}$ has countable index in H.

Take $h \in \mathrm{Stab_H(U)}$. For $u \in \psi(U)$, we see that $\psi^{-1}(h\psi(u)h^{-1}) \in U$. Hence, $h\psi(u)h^{-1} \in \psi(U)$, so $\mathrm{Stab_H(U)} \leq \mathrm{N_H(\psi(U))}$. The group $N_H(\psi(U))$ thus has countable index in H, and via the Baire category theorem, it follows that $N_H(\psi(U))$ is open in H.

Put $L := N_H(\psi(U))$. Given a coset kU and $l \in U$,

$$l.(kU) = \psi^{-1}(l\psi(k)\psi(U)l^{-1}) = \psi^{-1}(l\psi(k)l^{-1}\psi(U)) = k'U$$

for some $k' \in G$. We thus obtain an action of L on $\{kU \mid k \in G\}$. There are only countably many left cosets kU of U in G, so $\mathrm{Stab_L(gU)}$ has countable index in L.

As in the second paragraph, $N_L(\psi(gU))$ has countable index in L and is closed. Hence, $N_L(\psi(gU))$ is open in L, and so, $N_H(\psi(gU))$ is open in H. □

Proposition 4.33. *If G and H are t.d.l.c. Polish groups and $\psi : G \to H$ is a normal compression, then the ψ-equivariant action is continuous.*

Proof. Let $\alpha : H \times G \to G$ by $(h, g) \mapsto h.g$ be the action map. The topology on G has a basis consisting of cosets of compact open subgroups. It thus suffices to show $\alpha^{-1}(kU)$ is open for any $k \in G$ and $U \in \mathcal{U}(G)$.

Fix $k \in G$ and $U \in \mathcal{U}(G)$ and let $(h, g) \in \alpha^{-1}(kU)$. Lemma 4.29 ensures the map ψ_h is continuous. There is then $W \in \mathcal{U}(G)$ such that $h.(gW) = \phi_h(gW) \subseteq kU$. Additionally, Lemma 4.32 tells us that $L := N_H(\psi(gW))$ is open.

We now consider the open neighborhood $hL \times gW$ of (h, g). For $(hl, gw) \in hL \times gW$,

$$\begin{aligned} \alpha(hl, gw) = hl.gw &= \psi^{-1}(hl\psi(g)\psi(w)l^{-1}h^{-1}) \\ &= \psi^{-1}(h\psi(g)\psi(w')h^{-1}) \\ &= h.(gw'). \end{aligned}$$

The element $h.(gw')$ in kU, so $\alpha(hl, gw) \in kU$. We conclude that $\alpha(hL \times gW) \subseteq kU$, and thus α is continuous. □

In view of Proposition **??**, Proposition 4.33 allows us to conclude the semidirect product $G \rtimes H$ is a t.d.l.c. Polish group. To emphasize the ψ-equivariant action, we denote this semidirect product by $G \rtimes_\psi H$. If $O \leq H$ is a subgroup, we can form the semi-direct product $G \rtimes_\psi O$ by restricting the action of H to O.

Our next theorem gives a natural factorization of a normal compression.

Theorem 4.34. *Suppose that G and H are t.d.l.c. Polish groups and $\psi : G \to H$ is a normal compression. For $U \leq H$ an open subgroup, the following hold:*

(1) $\pi : G \rtimes_\psi U \to H$ via $(g, u) \mapsto \psi(g)u$ is a continuous surjective homomorphism with $\ker(\pi) = \{(g^{-1}, \psi(g)) \mid g \in \psi^{-1}(U)\}$;

(2) $\psi = \pi \circ \iota$ *where* $\iota : G \to G \rtimes_\psi U$ *is the usual inclusion; and*

(3) $G \rtimes_\psi U = \overline{\iota(G)\ker(\pi)}$*, and the subgroups* $\iota(G)$ *and* $\ker(\pi)$ *are closed normal subgroups of* $G \rtimes_\psi U$ *with trivial intersection.*

Proof. (1) The image of π is $\psi(G)U$. As $\psi(G)$ is dense and U is an open subgroup, it follows that $\psi(G)U = H$. Hence, π is surjective. By definition,

$$(g, u)(g', u') = (g \cdot u.g', uu').$$

In view of Lemma 4.31, we see that

$$\begin{aligned}\pi(g \cdot u.g', uu') = \psi(g \cdot (u.g'))uu' &= \psi(g)u\psi(g')u^{-1}uu' \\ &= \psi(g)u\psi(g')u' \\ &= \pi(g, u)\pi(g', u').\end{aligned}$$

Hence, π is a homomorphism. To see that π is continuous, it suffices to check that π is continuous at 1. Take $V \leq U$ a compact open subgroup of H. The preimage $\pi^{-1}(V)$ contains $\psi^{-1}(V) \times V$ which is an open neighborhood of 1. Hence, π is continuous at 1. Finally, an easy calculation shows $\ker(\pi) = \{(g^{-1}, \psi(g)) \mid g \in \psi^{-1}(U)\}$. Claim (1) is thus demonstrated.

Claim (2) is immediate.

(3) By Claim (1), $\iota(G) = \{(g, 1) \mid g \in G\}$ intersects $\ker(\pi)$ trivially, and both $\iota(G)$ and $\ker(\pi)$ are closed normal subgroups. The product $\iota(G)\ker(\pi)$ is dense, since it is a subgroup containing the set $\{(1, h) \mid h \in \psi(G) \cap U\} \cup \iota(G)$. We have thus verified (3). □

The factorization established in Theorem 4.34 allows us to make statements about the relationship between normal subgroups of G and H, when there is a normal compression $\psi : G \to H$. These results will be essential in establishing the key refinement theorem.

Proposition 4.35. *Let* G *and* H *be t.d.l.c. Polish groups,* $\psi : G \to H$ *be a normal compression, and* K *be a closed normal subgroup of* G*.*

(1) The image $\psi(K)$ *is a normal subgroup of* H*.*

(2) If $\psi(K)$ *is also dense in* H*, then* $\overline{[G, G]} \leq K$*, and every closed normal subgroup of* K *is normal in* G*.*

Proof. Form the semidirect product $G \rtimes_\psi H$, let $\iota : G \to G \rtimes_\psi H$ be the usual inclusion, and let $\pi : G \rtimes_\psi H \to H$ be the map given in Theorem 4.34.

(1) The intersection $\ker(\pi) \cap \iota(G)$ is trivial, so $\ker(\pi)$ centralizes $\iota(G)$, since each of the groups is normal in $G \rtimes_\psi H$. In particular, $\ker(\pi)$ normalizes $\iota(K)$. The normalizer $N_{G \rtimes_\psi H}(K)$ therefore contains the dense subgroup $\iota(G)\ker(\pi)$. As $\iota(K)$ is a closed subgroup of $G \rtimes_\psi H$, we conclude that $\iota(K) \trianglelefteq G \rtimes_\psi H$, since normalizers of closed subgroups are closed. Theorem 4.34 now ensures that $\pi(\iota(K)) = \psi(K)$ is normal in H.

(2) Since π is a quotient map and $\psi(K) = \pi(\iota(K))$ is dense in H, it follows that $\iota(K)\ker(\pi)$ is dense in $G \rtimes_\psi H$. Claim (1) implies that $\iota(K)$ is a closed normal subgroup of $G \rtimes_\psi H$. The image of $\ker(\pi)$ is thus dense under the usual projection $\chi : G \rtimes_\psi H \to G \rtimes_\psi H/\iota(K)$. On the other hand, Theorem 4.34 ensures $\iota(G)$ and $\ker(\pi)$ commute, hence $\iota(G)/\iota(K)$ has dense centralizer in $G \rtimes_\psi H/\iota(K)$. The group $\iota(G)/\iota(K)$ is then central in $G \rtimes_\psi H/\iota(K)$, so in particular, $\iota(G)/\iota(K)$ is abelian. We conclude that G/K is abelian and $\overline{[G,G]} \leq K$.

Let M be a closed normal subgroup of K. The map $\psi \restriction_K : K \to M$ is a normal compression map. Applying part (1) to the compression map $\psi \restriction_K$, we see that $\psi(M)$ is normal in H, so in particular $\psi(M)$ is normal in $\psi(G)$. Since ψ is injective, M is in fact normal in G. □

For G a topological group and A a group acting on G by automorphisms, say that G is A**-simple** if A leaves no proper non-trivial closed normal subgroup of G invariant. For example, G is $\{1\}$-simple if and only if G is topologically simple.

Theorem 4.36. *Suppose that G and H are non-abelian t.d.l.c. Polish groups with $\psi : G \to H$ a normal compression and suppose that G and H admit actions by topological group automorphisms of a (possibly trivial) group A such that ψ is A-equivariant.*

(1) If G is A-simple, then so is $H/Z(H)$, and $Z(H)$ is the unique largest proper closed A-invariant normal subgroup of H.

(2) If H is A-simple, then so is $\overline{[G,G]}$, and $\overline{[G,G]}$ is the unique smallest non-trivial closed A-invariant normal subgroup of G.

Proof. (1) Let L be a proper closed normal A-invariant subgroup of H. Clearly, $\psi(G) \not\leq L$, so $\psi^{-1}(L)$ is a proper closed normal A-invariant subgroup of G and hence is trivial. The subgroups $\psi(G)$ and L are then normal subgroups of H with trivial intersection, so $\psi(G)$ and L commute. Since $\psi(G)$ is dense in H and centralizers are closed, $L \le Z(H)$. In particular, $H/Z(H)$ does not have any proper non-trivial closed normal A-invariant subgroup, and (1) follows.

(2) The subgroup $L := \overline{[G,G]}$ is preserved by every topological group automorphism of G and hence is normal and A-invariant; note $L \neq \{1\}$, since G is not abelian. The image $\psi(L)$ is therefore a non-trivial A-invariant subgroup of H and hence is dense. Proposition 4.35 further implies any closed A-invariant normal subgroup of L is normal in G.

Letting K be an arbitrary non-trivial closed A-invariant normal subgroup of G, Proposition 4.35 ensures the group $\psi(K)$ is normal in H. Since ψ is A-equivariant, $\psi(K)$ is indeed an A-invariant subgroup of H, so by the hypotheses on H, the subgroup $\psi(K)$ is dense in H. Applying Proposition 4.35 again, we conclude that $K \geq L = \overline{[G,G]}$. The subgroup L is thus the unique smallest non-trivial closed A-invariant normal subgroup of G, and (2) follows. $\square$

4.5.2 The proof

Lemma 4.37. *For K and L closed normal subgroups of a topological group G, the map $\phi : K/(K\cap L) \to \overline{KL}/L$ via $k(K\cap L) \mapsto kL$ is a G-equivariant normal compression map, where G acts on each group by conjugation.*

Proof. Exercise 4.13 $\square$

Lemma 4.38. *Let K_1/L_1 and K_2/L_2 be closed normal factors of a topological group G and let G act on each factor by conjugation. If $\psi : K_1/L_1 \to K_2/L_2$ is a G-equivariant normal compression, then $\mathrm{C}_G(K_2/L_2) = \mathrm{C}_G(K_1/L_1)$.*

Proof. Exercise 4.14. $\square$

Proof of Theorem 4.24. We leave the uniqueness of i to the reader in Exercise 4.15.

For existence, let $\alpha : G \to \mathrm{Aut}(K/L)$ be the homomorphism induced by the conjugation action of G on K/L. Since K/L is centerless, the normal subgroup $\mathrm{Inn}(K/L)$ is isomorphic as an abstract group to K/L. Every non-trivial subgroup of $\alpha(G)$ normalized by $\mathrm{Inn}(K/L)$ also has non-trivial intersection with $\mathrm{Inn}(K/L)$, since the centralizer of $\mathrm{Inn}(K/L)$ in $\alpha(G)$ is trivial.

Take i minimal such that $G_{i+1} \not\leq \mathrm{C}_G(K/L)$. The group $\alpha(G_{i+1})$ is then non-trivial and normalized by $\mathrm{Inn}(K/L)$, so $\mathrm{Inn}(K/L) \cap \alpha(G_{i+1})$ is non-trivial. Set

$$B := \mathrm{C}_{G_{i+1}}(K/L),\ R := \alpha^{-1}(\mathrm{Inn}(K/L)) \cap G_{i+1}, \text{ and } A := \overline{[R,K]B}.$$

The groups A and B are closed normal subgroups of G such that $G_i \leq B \leq A \leq G_{i+1}$.

Since $\mathrm{Inn}(K/L) \cap \alpha(G_{i+1})$ is non-trivial, there are non-trivial inner automorphisms of K/L induced by the action of R, so $\overline{[R,K]} \not\leq L$. Since K/L is a chief factor of G, it must be the case that $K = \overline{[R,K]L}$. If A/B is abelian, then $[A,A] \leq C_G(K/L)$, so $[[R,K],[R,K]]$ centralizes K/L. As K/L is topologically perfect, it follows that K/L has a dense center, so K/L is abelian, which is absurd. The closed normal factor A/B is thus non-abelian.

Set $C := \mathrm{C}_G(K/L)$ and $M := \overline{KC}$. We see that $K \cap C = L$ since K/L is centerless and that $A \cap C = B$ from the definition of B. As $K = \overline{[R,K]L}$, it is also the case that $\overline{AL} = \overline{KB}$, and thus,

$$M = \overline{KC} = \overline{KBC} = \overline{ALC} = \overline{AC}.$$

We are now in position to apply Lemma 4.37 and thereby obtain G-equivariant normal compression maps $\psi_1 : K/L \to M/C$ and $\psi_2 : A/B \to M/C$.

Lemma 4.31 implies that $\mathrm{C}_G(M/C) = \mathrm{C}_G(K/L) = C$, so M/C is centerless. The factor K/L is chief, and thus, it has no proper G-invariant closed normal subgroups. Theorem 4.36 ensures that M/C also has no G-invariant closed normal subgroups; that is to say, M/C is a chief factor of G. Applying Theorem 4.36 to ψ_2, the group $D := \overline{[A,A]B}$ is such that D/B is the unique smallest non-trivial closed G-invariant subgroup of A/B. In particular, D/B is a chief factor of G.

The map ψ_2 restricts to a G-equivariant compression from D/B to M/C, so $\mathrm{C}_G(D/B) = \mathrm{C}_G(M/C)$. Since M/C is non-abelian, D/B is also non-abelian. We conclude that $\mathrm{C}_G(D/B) = \mathrm{C}_G(K/L)$, and hence D/B is a non-abelian chief factor of G associated to K/L with $G_{i+1} \leq B < D \leq G_i$. The proof is now complete. □

Notes

The first hints of the essentially chief series seem to appear in the work of V.I. Trofimov [14]. Moreover, in loc. cit., Trofimov makes the crucial observation that quotienting a Cayley–Abels graph by a normal subgroup can only drop the degree. Independently and rather later, Burger–Mozes analyze the normal subgroups of certain t.d.l.c. groups acting on trees in [3] and in particular find minimal non-trivial closed normal subgroups. In [4], Caprace–Monod push parts of the analysis of Burger–Mozes much further. Finally, Reid and the author complete the story in [13].

Theorem 4.24 in fact holds for all Polish groups. We restrict our attention to the case of t.d.l.c. Polish groups to avoid appealing to facts from descriptive set theory. The interested reader can find the general statement and proof of Theorem 4.24 in [12].

4.6 Exercises

Exercise 4.1. Verify that the origin map and the edge reversal maps are well-defined in a quotient graph.

Exercise 4.2. Give a complete proof of Lemma 4.9.

Exercise 4.3. Show that the actions defined in Lemma 4.9 respect the origin and edge reversal maps.

Exercise 4.4. Let Γ be a locally finite graph and $G \leq \mathrm{Aut}(\Gamma)$ a closed subgroup. Fix $v \in V\Gamma$ and set $X := E(v)$. Show the homomorphism $\alpha : G_{(v)} \to \mathrm{Sym}(X)$ induced by the action of $G_{(v)}$ on X is continuous when $\mathrm{Sym}(X)$ is equipped with the discrete topology.

Exercise 4.5. Let G be a group acting on a graph Γ and suppose that $M \leq N$ are normal subgroups of G. Show there is a G-equivariant graph isomorphism between $(\Gamma/M)/(M/N)$ and Γ/N.

Exercise 4.6. Suppose that K/L is closed normal factor of a topological group G. Show $\overline{[K/L, K/L]} = \overline{[K, K]L}/L$.

Exercise 4.7. Let G be a topological group and K/L be a closed normal factor of G. Suppose that $D \subseteq G$ is such that DL/L is a dense subset of K/L. Show that if $g \in G$ is such that $[g, d] \in L$ for all $d \in D$, then $g \in \mathrm{C}_G(K/L)$.

Exercise 4.8. Let G be a group with K/L a closed normal factor of G. Taking $\pi : G \to G/L$ to be the usual projection, show $\mathrm{C}_G(K/L) = \pi^{-1}(\mathrm{C}_{G/L}(K/L))$.

Exercise 4.9. Let G and H be groups with $\psi : G \to H$ an injective homomorphism such that $\psi(G)$ is normal in H. Show the map $\phi_h : G \to G$ defined by $\phi_h(g) := \psi^{-1}(h\psi(g)h^{-1})$ is group automorphism of G for any $h \in H$.

Exercise 4.10. Let $\psi : G \to H$ be a normal compression of topological groups. Verify that the ψ-equivariant action is a group action of G on H.

Exercise 4.11. Let G and H be t.d.l.c. Polish groups with $\psi : G \to H$ a normal compression. Show every closed normal subgroup of G is invariant under the ψ-equivariant action of H on G.

Exercise 4.12. Prove Lemma 4.31.

Exercise 4.13. Prove Lemma 4.37.

Exercise 4.14. Prove Lemma 4.38.

Exercise 4.15. Verify the uniqueness claim of Theorem 4.24.

Exercise 4.16 (Trofimov; Möller)**.** For G a t.d.l.c. group, recall from Exercise 3.15 that $B(G) = \{g \in G \mid \overline{g^G} \text{ is compact}\}$, where g^G is the conjugacy class of g in G. Show $B(G)$ is closed for G a compactly generated t.d.l.c. group. **HINT:** Use Exercise 3.15 and Theorem 4.17.

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