Zero-Dimensional Symmetry and its Ramifications

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Outline

Zero-Dimensional Symmetry

- Symmetry and groups
- Symmetries of zero-dimensional structures
- Symmetry groups which are zero-dimensional

Ramifications

- Abstract structure
- Geometry
- Commensurators, local structure
- Computations and linear representations

What is symmetry?

Symmetry occurs when

- one part of a structure looks like another,
- or there is a repeating pattern,
- or when one part is interchangeable with another.

Symmetry can be

– visual

– auditory

– tactile

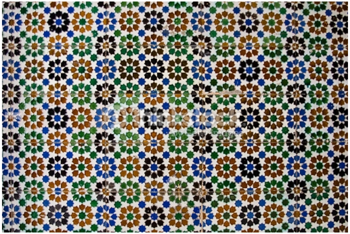
– mathematical

$$x + y + z + xyz$$

Symmetry occurs

– in nature

– in art



– and/or both

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

What is the mathematics of symmetry?

Symmetry occurs when

- one part of a structure looks like another,
- or there is a repeating pattern,
- or when one part is interchangeable with another.

Transformations of the structure which leave it looking unchanged form a *group* under composition.

The size of a structure is measured by assigning a number to it. Its symmetry is gauged by finding the group of transformations preserving its structure.

Different classes of groups account for symmetries of different types of structures.

Continuous symmetry

Smooth structures may have symmetry groups which allow one point to be moved continuously to another.

Figure: The group of rotations and reflections of the sphere is $O(3)$

Discrete symmetry

Seventeen types of wallpaper symmetry

Symmetry of relational structures

0-dimensional structures: simple graphs $\Gamma = (V(\Gamma), E(\Gamma))$

Symmetries: graph automorphisms

$G = \mathrm{Aut}(\Gamma)$ denotes the automorphism group. The *permutation topology $\mathfrak{T}$ on G* has the base

$$\mathfrak{N} := \left\{ N(g; F) \mid g \in G,\ F \underset{\text{finite}}{\subset} V(\Gamma) \right\}$$

with $N(g; F) := \{h \in G \mid h.v = g.v,\ \forall v \in F\}$.

$(G, \mathfrak{T})$ is a topological group.

Automorphisms of locally finite graphs

Suppose that Γ is $\boxed{\text{locally finite and connected}}$. Then G_v is open for each $v \in V(\Gamma)$ and is compact because

$$G_v \underset{\text{closed}}{\leq} \prod_{n \geq 0} \mathrm{Sym}(B(v, n)).$$

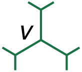
Hence $G = \mathrm{Aut}(\Gamma)$ is locally compact.

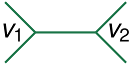
In fact, G_v is either finite or homeomorphic to the Cantor set. Hence G is *totally disconnected*.

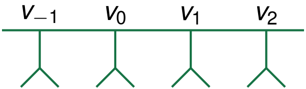
If Γ is also $\boxed{\text{vertex-transitive}}$, then G is *compactly generated*.

Automorphisms of locally finite graphs

Examples

- Let $\Gamma = Z^d$, the d-dimensional integer lattice. Then G_v is finite and G is discrete for each $d \geq 1$.
- Let $\Gamma = T_{q+1}$, the regular tree in which each vertex has valency $q + 1$. There are three types of automorphism:
  - *elliptic* 

  - *inversion* 

  - *hyperbolic* 

  When $q \geq 2$, G_v is infinite and G non-discrete.

Scale

Totally disconnected locally compact groups

Theorem

Every totally disconnected, locally compact topological space is 0*-dimensional.*

Theorem (van Dantzig, 1930s)

Let G be a t.d.l.c. group and $\mathscr{O}$ be a neighbourhood of the identity. Then there is a compact, open subgroup $U \subseteq \mathscr{O}$.

Corollary

Every compact t.d. group is profinite, i.e., is a projective limit of finite groups.
Conversely, every profinite group is compact and t.d.

Cayley-Abels graphs

Definition

Let G be a t.d.l.c. group and let $U \underset{\text{cpt, open}}{\leq} G$. Suppose that $G = \langle X \rangle$ with X compact, $X^{-1} = X$ and $X = UXU$ and put

$$V = G/U \text{ and } E = \{(gU, gxU) \mid g \in G,\ x \in X,\ gxU \neq gU\}.$$

Then $\Gamma(G; X, U) := \Gamma(V, E)$ is a *Cayley-Abels graph* for G.

Theorem

The Cayley-Abels graph $\Gamma(G; X, U)$ is locally finite and $G \curvearrowright \Gamma$ by graph automorphisms. The homomorphism $G \to Aut(\Gamma)$ has closed image and compact kernel.

Four themes of research

Interrelated themes and approaches to the research.

Abstract structure T.d.l.c. groups compared through homomorphisms and decompositions. Special subgroups.

Geometry Actions of t.d.l.c. groups on graphs and more general structures elucidate the global structure of the groups.

Local structure Locally profinite properties may distinguish different t.d.l.c. groups. Commensurated subgroups.

Computation Methods for computation and linear representations of t.d.l.c. groups.

Constructing groups

The following constructions illustrate and motivate definitions and results to follow.

Examples

- Let F be a finite group. For each $N \in \mathbb{Z}$, put

$$G_N = \left\{ g \in F^{\mathbb{Z}} \mid g_n = 1 \text{ if } n < N \right\}.$$

  Then G_N is profinite and $G_N \underset{\text{open}}{\leq} G_M$ if $M \leq N$.

  Define $G = \bigcup_{N \in \mathbb{Z}} G_N$.
  Then G is a t.d.l.c. group and $G_N \triangleleft G$ for each N.
- Define the *shift*, $\alpha \in \mathrm{Aut}(G)$, by $\alpha(g)_n = g_{n+1}$.
  Then $G \rtimes_\alpha \mathbb{Z}$ is a t.d.l.c. group which has no compact open normal subgroups.

Scale

Construction of t.d.l.c. groups from basic cases

Definition (P. Wesolek)

The class of *elementary groups* is the smallest class $\mathscr{E}$ of t.d.l.c.s.c. groups closed under the operations illustrated in the Example: contains all profinite and discrete groups, increasing unions; extensions; (plus) subgroups and quotients.

Wesolek defines a *decomposition rank* of G: a countable ordinal that measures the number of steps needed to build G from the basic ingredients of profinite and discrete groups.

Examples

Compactly generated (virtually) topologically simple t.d.l.c. groups, such as $\mathrm{Aut}(T_{q+1})$ and $PSL_n(\mathbb{Q}_p)$, are not elementary.

Deconstruction into normal subgroups and quotients

If a group G has a normal subgroup N, then G factors into N and the quotient group G/N. In classes of groups such as finite groups and Lie groups, N and G/N are smaller and repeated factoring terminates after a finite number of steps. Groups which cannot be factored are called *simple*.

That is not the case for t.d.l.c. groups. However ...

Theorem

Let G be a compactly generated t.d.l.c. group.

(P.-E. Caprace, N. Monod) *Either: G has an infinite discrete quotient; or G has a co-compact normal subgroup having a finite number of non-compact simple quotients.*

(C. Reid, P. Wesolek) *G admits an essentially chief series.*

The scale and special subgroups

Definition

The *scale* of an endomorphism $\alpha : G \to G$ is the positive integer

$$s(\alpha) = \min\Big\{[\alpha(U) : \alpha(U) \cap U] \mid U \underset{\text{cpt, open}}{\leq} G\Big\}.$$

U is *minimising* if the minimum is attained at U.

General t.d.l.c. groups are far from being Lie groups but:

- the scale of α can play the role of eigenvalues in the adjoint representation of a Lie group
- minimising subgroups correspond to canonical forms in many examples.

The structure of minimising subgroups

Theorem

Let $\alpha \in End(G)$ and $U \leq G$ be compact and open. Define

$$U_+ = \{u \in U \mid \exists (u_n)_{n\geq 0} \subset U \text{ with } u_0 = u \text{ and } \alpha(u_{n+1}) = u_n\}$$
$$U_- = \{u \in U \mid \alpha^n(u) \in U \text{ for all } n \geq 0\}.$$

Then U is minimising for α if and only if

- TA $U = U_+U_-$ and
- TB $U_{--} := \bigcup_{n\geq 0} \alpha^{-n}(U_-)$ is closed.

Definition

A compact open subgroup satisfying TA and TB is *tidy for α*.

The scale and tidy subgroups

Examples

- The shift automorphism has:
  $s(\alpha) = |F|$, $U = G_0$, $U_+ = G_0$ and $U_- = U_{--} = \{1\}$;
  $s(\alpha^{-1}) = 1$, $U = G_0$, $U_+ = \{1\}$, $U_- = G_0$ and $U_{--} = G$.
- $G = \mathrm{Aut}(T_{q+1})$, the inner automorphism α_g has:
  g elliptic $s(\alpha_g) = 1$, $U = G_v = U_+ = U_-$;
  g inversion $s(\alpha_g) = 1$, $U = G_{v_1} \cap G_{v_2} = U_+ = U_-$; and
  g hyperbolic $s(\alpha_g) = q^s$, $U = G_{v_0} \cap G_{v_1}$, $U_+ = \bigcap_{n=0}^{\infty} G_{v_n}$
  $U_- = \bigcap_{n=-\infty}^{1} G_{v_n}$.
- $G = PSL_n(\mathbb{Q}_p)$, α is conjugation by $\mathrm{diag}(1, p \dots, p^{n-1})$ has:
  $s(\alpha) = p^{(n^3-n)/6}$, U is the Iwahori subgroup, U_+ is the subgroup of upper triangular matrices, U_- the subgroup of lower triangular matrices.

Special subgroups

- Suppose that G is a t.d.l.c. group, that $x \in G$ and that V is tidy for x. Then $V_{++} \rtimes \langle x \rangle$ is a closed subgroup of G. These subgroups are analogues of $(ax + b)$-groups in Lie groups.
- The *contraction subgroup* for $\alpha \in \mathrm{Aut}(G)$ is $\mathit{con}(\alpha) = \{x \in G \mid \alpha^n(x) \to 1 \text{ as } n \to \infty\}$.
  (Glöckner & W.) Structure of $\mathit{con}(\alpha)$ when closed.
- $\mathit{nub}(\alpha) = \bigcap \{V \text{ tidy for } \alpha\}$. $\mathit{con}(\alpha)$ is closed if and only if $\mathit{nub}(\alpha)$ is trivial.
- Analogues of parabolic and Levi subgroups.

Flatness

Definition

Let G be a t.d.l.c. group. The subgroup $\mathcal{H} \leq \mathrm{Aut}(G)$ is *flat* if there is $U \underset{\text{cpt open}}{\leq} G$ that is tidy for every $\alpha \in \mathcal{H}$.

Theorem

1. *Every fin. gen. nilpotent subgroup of $Aut(G)$ is flat.*
2. *Suppose that $\mathcal{H} \leq Aut(G)$ is fin. gen. and flat. Let U be tidy for $\mathcal{H}$. Then $\mathcal{H}/N_{\mathcal{H}}(U)$ is free abelian and*

$$U = U_0 U_1 \dots U_s,$$

with either $\alpha(U_j) \leq U_j$ or $\alpha(U_j) \geq U_j$ for every $\alpha \in \mathcal{H}$.

Example

Diagonal matrices induce a flat subgroup of $\mathrm{Aut}(PSL_n(\mathbb{Q}_p))$.

Can we do better than the Cayley-Abels graph?

The Cayley-Abels graph $\Gamma(G; X, U)$ is not unique, it varies with the choice of X and U. It does not record precise information about G.

The *Tits building* is a geometry that, for special cases such as simple linear groups, records precise information about G. For example, it is a simplicial complex and its simplices correspond to the maximal open profinite subgroups of G.

Finding a canonical geometric representation for a general simple group appears to be beyond reach but it might be possible to do better if the group has rank greater than 1.

In the case of rank 1 linear groups, the Tits building is an infinite tree. Tits also studied automorphism groups of trees and these are sources of many rank 1 examples.

Automorphism groups of trees

Groups acting on trees are an essential part of the structure of general t.d.l.c. groups.

Theorem (Tree Representation, Baumgartner & W.)

Let $\alpha \in Aut(G)$ and suppose that $s(\alpha) > 1$.
Then $V_{++} \rtimes \langle\alpha\rangle$ has a proper action on the tree $T_{s(\alpha)+1}$ and this action fixes an end of the tree.

The groups $V_{++} \rtimes \langle\alpha\rangle$ acting on regular trees correspond to self-similar groups acting on rooted trees, which are studied in geometric group theory.
Current work aims to understand these groups.

Local structure

The theorem of van Dantzig ensures that every t.d.l.c. group has a compact open subgroup.
Suppose that U, V are compact open subgroups of G. Then

$$[U : U \cap V] \text{ and } [V : U \cap V] < \infty,$$

that is, U and V are *commensurable*.
By *local structure* of G is meant properties of the commensurability class of compact open subgroups of G.

Definition

A compact subgroup $K \leq G$ is *locally normal* if $N_G(K)$ is open.
$\mathcal{LN}(G)$ denotes the set of commensurability classes of locally normal subgroups of G.

The structure lattice

When G is a p-adic Lie group, locally normal subgroups of G correspond to ideals in the Lie algebra of G.

In general, $\mathcal{LN}(G)$ is a modular lattice and is called the *structure lattice*.

$\mathcal{LN}(G)$ may be used to distinguish different types of local structure for topologically simple compactly generated t.d.l.c. groups (P.-E. Caprace, C. Reid and W.).
There are possibly five different types but it is not known whether all five types are populated.

The structure lattice

Two of the types of local structure of simple t.d.l.c. groups are: *locally decomposable*, for example $\mathrm{Aut}(T_{q+1})$; and *locally h.j.i.*, (*i.e.* $\mathcal{LN}(G) = \{0, 1\}$), for example $PSL(n, \mathbb{Q}_p)$.

Theorem (P.-E. Caprace, C. Reid & W.)

Let G be a topologically simple t.d.l.c. group. Then

- *G has finite local prime content, and all locally normal subgroups have the same local prime content; and*
- *if G is locally decomposable, then it is not amenable.*

Stretch target: Computing in t.d.l.c. groups

T.d.l.c. groups are topological groups and so computations are only carried out up to approximation. One approach to a ‘classification’ might be to describe classes of groups to which the same approximation methods apply.

- Computations in matrix groups can be done using abstract algebra followed by approximating the matrix entries *e.g.* p-adic numbers. (Numerical linear algebra.)
- Graph automorphisms are approximated by describing their action on large finite sets. When can this be done systematically? First step: describe large finite sets for drawing algorithms.
- Concrete goal Develop algorithms which apply in large classes of groups for computing the scale.

Linear representations of t.d.l.c. groups

The representation theory of linear groups over p-adic fields is a significant area of research. Less is known about unitary representations of other t.d.l.c. groups.

- Describe the unitary representations of $V_{++} \rtimes \langle \alpha \rangle$.
  How much information about G does $C^*(G)$ remember?
  Describe the Hecke algebra (G, V_+).
- Structures important in the representation theory of p-adic Lie groups have been abstracted in terms related to tidy subgroups. Describe groups having these structures.

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