Advances in Index Theory

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INSTITUTE FOR GEOMETRY AND ITS APPLICATIONS

THE UNIVERSITY of ADELAIDE

Australian Government
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2017 Australian Laureate Fellowships

Professor Mathai Varghese

Administering Organisation	The University of Adelaide
Discipline Area	Mathematics, Physics, Chemistry and Earth Sciences

Fellowship project summary

Advances in index theory and applications (FL170100020)

The project aims to develop novel techniques to investigate Geometric analysis on infinite dimensional bundles, as well as Geometric analysis of pathological spaces with Cantor set as fibre, that arise in models for the fractional quantum Hall effect and topological matter, areas recognised with the 1998 and 2016 Nobel Prizes. Building on the applicant's expertise in the area, the project will involve postgraduate and postdoctoral training in order to enhance Australia's position at the forefront of international research in Geometric Analysis. Ultimately, the project will enhance Australia's leading position in the area of Index Theory by developing novel techniques to solve challenging conjectures, and mentoring HDR students and ECRs.

Australian Research Council funding: $1,638,060 + $500,000

About Professor Varghese

Professor Mathai Varghese is the Director of The University of Adelaide's Institute for Geometry and its Applications, Elder Professor of Mathematics in the School of Mathematical Sciences and an Adjunct Professor of Mathematics within the Mathematical Sciences Institute at The Australian National University. Professor Varghese is a major contributor to the geometric analysis field, and is internationally renowned for the Mathai-Quillen formalism in topological field theories, and several other seminal articles. Professor Varghese has fellowships at the Australian Academy of Science (2011), the Australian Mathematical Society (2000) and the Royal Society of South Australia (2013). Professor Varghese was awarded the Australian Mathematical Society Medal in 2000

Find out more about Professor Varghese and his research by visiting his profile page on The University of Adelaide website. [1]

For further information about this funding scheme, please visit the *Australian Laureate Fellowships* scheme page on the ARC website [2].

References

[MM17]

V. M. and R.B. Melrose,

Geometry of Pseudodifferential algebra bundles

and Fourier Integral Operators,

Duke Mathematical Journal, 166 no.10 (2017) 1859-1922.

+ work in progress

Outline of talk

1. Glimplse of Atiyah-Singer index theorem and applications
2. Pseudodifferential operators
3. Fourier Integral operators (FIOs) as automorphisms of pseudodifferential operators.
4. Algebra bundles of pseudodifferential operators - construction
5. Examples of purely infinite dimensional algebra bundles of pseudodifferential operators.

The Atiyah-Singer Index Theorem

In the 1960s, **Sir Michael Atiyah** and **Isadore Singer** proved what was to become one of the most important and widely applied theorems in 20th century mathematics and continues to have significant impact to this day, namely the

Atiyah-Singer Index Theorem.

The laws of nature are often expressed in terms of (partial) differential equations, which if **elliptic**, have an

index = (# of solutions) — (# of constraints imposed).

The Atiyah-Singer Index Theorem gives a striking calculation of this **index** in terms of geometry and topology.

Abel Prize awarded by the Norwegian king (2004)

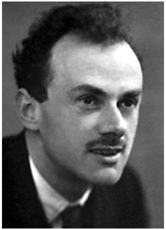
Atiyah-Singer were awarded the prestigious **Abel Prize** in 2004.

50^{th} birthday of the Atiyah-Singer Index Theorem,
. 1963 $\Rightarrow$ 2013

Michael F. Atiyah, Isadore M. Singer,
The index of elliptic operators on compact manifolds.
Bulletin of the American Mathematical Society **69** (1963) 422–433.

Dirac operators

Paul Dirac was one of the founders of quantum mechanics, and was awarded the **Nobel Prize in Physics** in 1933.



$$\boxed{\sqrt{-1}\hbar\gamma^\mu\partial_\mu\psi = mc\psi}$$

Dirac defined an operator $\partial\!\!\!/$ on $\mathbb{R}^n$ that solved the **square root problem** for the Laplacian on $\mathbb{R}^n$, that is, $\partial\!\!\!/^2 = \Delta$.

The construction was novel as it used **Clifford algebras and spinors** in an essential way.

Dirac operators

More precisely, if $\{\gamma_j\}_{j=1}^n$ denote Clifford multiplication by an orthonormal basis of $\mathbb{R}^n$, then the Clifford algebra relations are $\gamma_j\gamma_k + \gamma_k\gamma_j = 2\delta_{ij}$. When $n = 2$, these are **Pauli matrices**;

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\partial\!\!\!/ = \sum_{j=1}^{n} \gamma_j \frac{\partial}{\partial x_j}.$$

It turns out that this operator plays a fundamental role in quantum mechanics, and is known as the **Dirac operator**.

By construction,

$$\partial\!\!\!/^2 = \Delta.\mathrm{Id}.$$

The Index Theorem for Dirac operators

Atiyah & Singer extended the definition of the **Dirac operator**, $\partial\!\!\!/^{+}$ to any compact **spin** manifold Z of even dimension.

On such a manifold, there are **half spinor bundles** $\mathcal{S}^{\pm}$.

Locally, the Dirac operator is constructed similar to the case on Euclidean space,

$$\partial\!\!\!/ = \sum_{j=1}^{n} \gamma_j \nabla_{e_j}.$$

where ∇ is the induced Levi-Civita connection on $\mathcal{S}^{\pm}$. Then

$$\partial\!\!\!/^{+} : C^{\infty}(Z, \mathcal{S}^{+}) \to C^{\infty}(Z, \mathcal{S}^{-})$$

This operator has the property that

$$\partial\!\!\!/^{2} = \Delta + \frac{R}{4}$$

The Index Theorem for Dirac operators

This operator is elliptic, and Atiyah-Singer gave a striking computation of the analytic index,

$$\mathrm{Index}_a(\partial\!\!\!/^+) = \mathsf{dim}(\mathrm{nullspace}\,\partial\!\!\!/^+) - \mathsf{dim}(\mathrm{nullspace}\,\partial\!\!\!/^-)$$
$$= \int_Z \widehat{A}(Z) \quad \in \mathbb{Z}$$

where RHS is the A-hat genus of the manifold Z. In terms of the Riemannian curvature Ω_Z of Z,

$$\widehat{A}(Z) = \sqrt{\det\left(\frac{\frac{1}{4\pi}\Omega_Z}{\sinh(\frac{1}{4\pi}\Omega_Z)}\right)}$$

Examples of the Atiyah-Singer Index Theorem

Consider the simplest case of the circle S^1. Let

$$L^2(S^1) = \left\{ \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n\theta} \;\middle|\; \sum_{n \in \mathbb{Z}} |a_n|^2 < \infty \right\}$$

denote the Hilbert space of square integrable functions on S^1.

Consider the closed subspace of $L^2(S^1)$

$$H(S^1) = \left\{ \sum_{n \geq 0} a_n e^{2\pi i n\theta} \;\middle|\; \sum_{n \geq 0} |a_n|^2 < \infty \right\}$$

and let

$$P : L^2(S^1) \to H(S^1)$$

denote the orthogonal projection.

Examples of the Atiyah-Singer Index Theorem

Let $f : S^1 \to \mathbb{C}$ be a continuous function and denote by M_f the operator on $L^2(S^1)$ given by multiplication by f.

M_f is a bounded operator, bounded by $||f||_\infty = \mathrm{supremum}|f(\theta)|$.

Consider the operator $T_f : H(S^1) \to H(S^1)$ given by the composition

$$T_f = P \circ M_f$$

T_f is a bounded operator, bounded by $||f||_\infty$.

It turns out that T_f has an index (i.e. is Fredholm) whenever f is nowhere zero, i.e.

$$f : S^1 \to \mathbb{C} \setminus \{0\}$$

Examples of the Atiyah-Singer Index Theorem

On the other hand, given a continuous map $f : S^1 \to \mathbb{C} \setminus \{0\}$, it has a **winding number** or degree, $w(f)$ which intuitively is the number of times the map winds around the circle.

It is defined purely topologically as follows. The homomorphism induced on fundamental groups

$$f_* : \mathbb{Z} \cong \pi_1(S^1) \to \pi_1(\mathbb{C} \setminus \{0\}) \cong \mathbb{Z}$$

can be identified uniquely by an integer, defined to be $w(f)$.

Then the index theorem for the circle is

Toeplitz Index Theorem

$$\mathrm{index}(T_f) = -w(f)$$

Examples of the Atiyah-Singer Index Theorem

Now consider the case of a compact Riemann surface Σ.
The de Rham operator $d : \Omega^j(\Sigma) \to \Omega^{j+1}(\Sigma)$ for $j = 0, 1$ and its adjoint $d^* : \Omega^{j+1}(\Sigma) \to \Omega^j(\Sigma)$

Then

$$d + d^* : \Omega^0(\Sigma) \oplus \Omega^2(\Sigma) \to \Omega^1(\Sigma)$$

has an index, and the index theorem is

Gauss Bonnet theorem

$$\mathrm{index}(d + d^*) = 2 - 2g = \chi(\Sigma) = \frac{1}{2\pi} \int_\Sigma R \; d\mathrm{vol}$$

where g is the genus and $\chi(\Sigma)$ is the Euler characteristic of Σ.

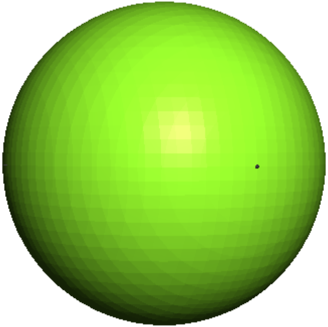
A rather simple consequence is the classification (or listing) of all 2-dimensional oriented surfaces, up to continuous deformation

Curvature of space

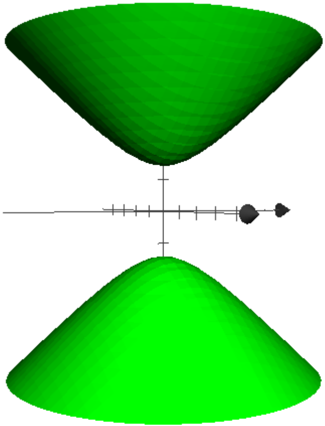
Curvature can be illustrated in the following slides.

Amongst these, positive curvature is probably most important, being intimately related to mass in General Relativity.

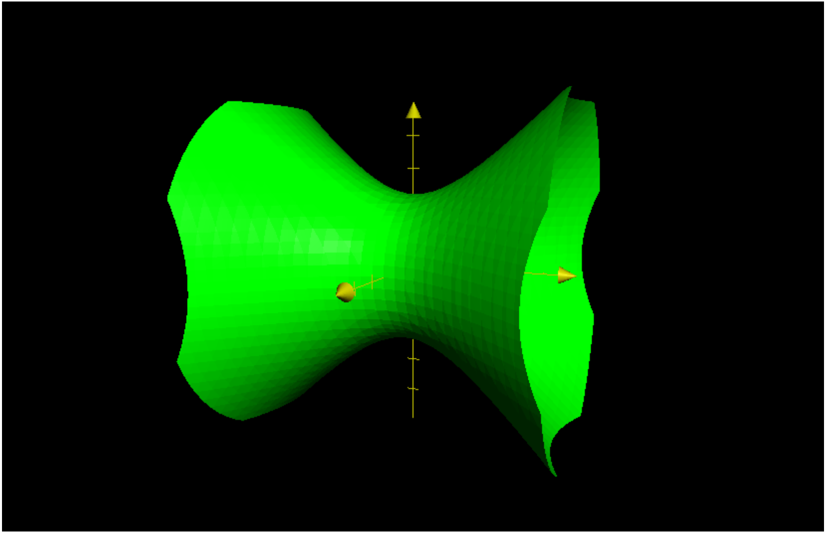
Positive Curvature



Negative Curvature



Saddle Curvature

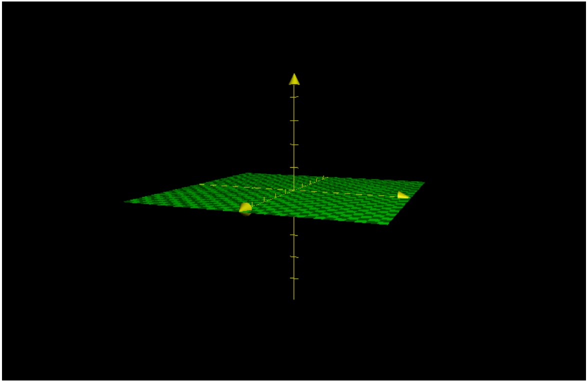


Application of Atiyah-Singer index theorem

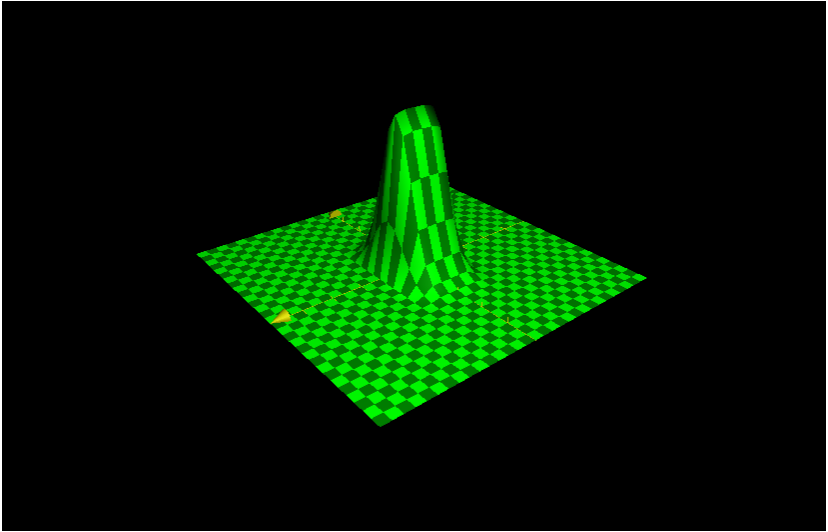
First application (of curvature)

The first application of the index theorem is a purely mathematical result that is a direct consequence of it, but may be somewhat surprising if seen for the first time.

Suppose we start off with the flat plane,

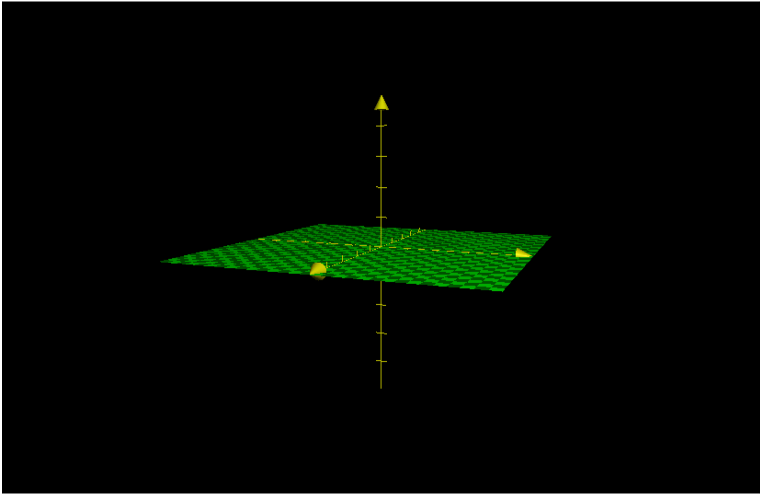


Rigidity of scalar curvature



and ask ourselves if it is possible to perturb the metric inside a disk in the plane such that the curvature is positive?

Rigidity of scalar curvature



The striking answer to this question is **no**! Any perturbation of the metric inside a disc has to be flat everywhere!

This property is called the **rigidity** of scalar curvature.

Rigidity of scalar curvature

The proof is a consequence of the Gauss Bonnet Theorem. Namely, graft this disk onto a large enough torus and get a contradiction using the Gauss Bonnet Theorem in genus 1.

$$0 = \chi(S^1 \times S^1) = \frac{1}{2\pi} \int_{S^1 \times S^1} R \, d\mathrm{vol} > 0$$

The rigidity of scalar curvature continues to be true for higher dimensional Euclidean spaces. However, one now has to use more sophisticated index theorems to conclude. This result is due to **Gromov-Lawson** (1981) via the families index theorem method, and to **Schoen-Yau** (1979) for dimensions ≤ 7 via minimal surfaces.

Exotic spheres

You can get any n-sphere by taking two n-dimensional balls and gluing them together along their boundary using some orientation-preserving diffeomorphism

$$f : S^{n-1} \to S^{n-1}.$$

Orientation-preserving diffeomorphisms like this form a group called $Diff_+(S^{n-1})$. Using the above trick, it turns out that the group of smooth structures on the n-sphere is isomorphic to the group of connected components of $Diff_+(S^{n-1})$,

$$\pi_0(Diff_+(S^{n-1})).$$

Exotic spheres

Exotic spheres

Smooth Poinjcaré Conjecture:

Every topological sphere is diffeomorphic to the standard sphere S^n.

The exotic spheres are counterexamples to this conjecture.

The first known and lowest dimensional exotic spheres are in dimension 7. In the late 1950s, Milnor defined an invariant, which now can be rephrased in terms of Atiyah-Singer index theory invariants, and showed that there were 28 seven dimensional spheres that were not diffeomorphic to each other.

dimension n	smooth structures on the n-sphere
1	1
2	1
3	1
4	??
5	1
6	1
7	$\mathbf{Z}/28$

Table 1: Exotic spheres

Work with Melrose: pseudodifferential bundles

Pseudodifferential operators and their symbols

Consider a differential operator P on an open set $\Omega \subset \mathbb{R}^n$ of order m with C^∞ coefficients,

$$P = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha, \qquad a_\alpha \in C^\infty(\Omega)$$

The **(full) symbol** of P is

$$\sigma(P)(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha$$

and the **principal symbol** of P is

$$\sigma_m(P)(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha$$

Pseudodifferential operators and their symbols

Using the Fourier transform composed with the inverse Fourier transform, one has the following oscillatory integral representation, for $u \in C_c^\infty(\Omega)$

$$Pu(x) = (2\pi)^{-n} \int\int e^{i(x-y).\xi} \sigma(P)(x,\xi) u(y) dy d\xi.$$

A **pseudodifferential operator** P is also of this form, where now the full symbol $\sigma(P)(x,\xi)$, has an asymptotic expansion:

$$\sigma(P)(x,\xi) \sim \sum_{j=0}^{\infty} p_{m-j}(x,\xi).$$

Here each term $p_{m-j}(x,\xi) \in C^\infty(\Omega \times \mathbb{R}^n \setminus \{0\})$ is homogenous, $p_{m-j}(x, t\xi) = t^{m-j} p_{m-j}(x,\xi)$ for $t > 0$ and $p_{m-j}(x,\xi) \in S^{m-j}(\Omega)$.

Pseudodifferential operators and their symbols

The **principal symbol** of P is $\sigma_m(P)(x,\xi) = p_m(x,\xi) \in S^m(\Omega)$ and the order of P is m. P is said to be **elliptic** if the principal symbol $\sigma_m(P)(x,\xi) \neq 0$ when $\xi \neq 0$.

The symbol algebra is **graded**, $S^m(\Omega).S^k(\Omega) \subset S^{m+k}(\Omega)$, but the algebra of pseudodifferential operators is only **filtered** as it turns out that $\Psi^m(\Omega)/\Psi^{m-1}(\Omega) \cong S^m(\Omega)$. The associated graded algebra,

$$\mathrm{Gr}(\Psi^\bullet(\Omega)) = S^\bullet(\Omega).$$

It turns out that all this also makes sense for compact manifolds Z and a vector bundle V over Z also and the notation is

$$\Psi^\bullet(Z,V)), \qquad S^\bullet(T^*Z, \pi^*\mathrm{End}\, V)$$

and once again one has, $\mathrm{Gr}(\Psi^\bullet(Z,V)) = S^\bullet(T^*Z, \pi^*\mathrm{End}\, V)$.

Atiyah-Singer index theorem: motivation

Atiyah and Singer generalized their famous index theorem to **families** of elliptic operators as follows. First consider the case of a smooth map $X \longrightarrow \Psi^{\mathbb{Z}}(Z, V)$, giving rise to a smooth family of pseudodifferential operators on Z parametrised by X. This is generalised as follows.

Consider a **fiber bundle of compact manifolds**

$$\begin{tikzcd} Z \arrow[r] & Y \arrow[d, "\phi"] \\ & X. \end{tikzcd}$$

and let $V \to Y$ be a vector bundle on the total space.

Let $\quad \Psi^{\mathbb{Z}} = \Psi^{\mathbb{Z}}(Y/X; V) \quad$ be the filtered algebra bundle over X, with typical fibre $\Psi^{\mathbb{Z}}(Y_x; V|_{Y_x}) \cong \Psi^{\mathbb{Z}}(Z; V|_Z), \quad x \in X$.

Atiyah-Singer index theorem: motivation

$$\begin{tikzcd} \Psi^{\mathbb{Z}}(Z; V|_Z) \arrow[r] & \Psi^{\mathbb{Z}} \arrow[d] \\ & X. \end{tikzcd}$$

Since the fibre bundle Y is locally trivial, $Y\Big|_U \cong U \times Z$ for contractible open subsets U of X, we see that $\Psi^{\mathbb{Z}}\Big|_U \cong U \times \Psi^{\mathbb{Z}}(Z; V|_Z)$ for contractible open subsets U of X, so that $\Psi^{\mathbb{Z}}$ is a locally trivial bundle. The structure group of $\Psi^{\mathbb{Z}}$ is the same as the structure group of Y, namely the diffeomorphism group $\mathrm{Diff}(Z)$ or more accurately $\mathrm{Aut}(V\Big|_Z)$.

Atiyah-Singer index theorem: motivation

If D is an elliptic section of the bundle $\Psi^{\mathbb{Z}} \to X$, then **Atiyah** and **Singer** showed that

$$\mathrm{index}(D) \in K^0(X)$$

is expressed in terms of the topology of Y, V and symbol of D.

Melrose and I consider general **filtered algebra bundles** $\Psi^{\mathbb{Z}} \to X$ with typical fibre $\Psi^{\mathbb{Z}}(Z; V)$, the filtered algebra of classical pseudodifferential operators on Z acting on sections of a vector bundle $V \to Z$.

If D is an elliptic section of $\Psi^{\mathbb{Z}} \to X$, then it turns out that

- $\mathrm{index}(D) \in K^0(X; \delta(\Psi^{\mathbb{Z}}))$, the twisted K-theory of X,
- $\delta(\Psi^{\mathbb{Z}}) \in H^3(X; \mathbb{Z})$ is the **Dixmier-Douady class**,

a characteristic class of the bundle $\Psi^{\mathbb{Z}}$.

Atiyah-Singer index theorem: motivation

The structure group of a filtered algebra bundle $\Psi^{\mathbb{Z}} \to X$ with typical fibre $\Psi^{\mathbb{Z}}(Z; V)$ is equal to $\mathrm{Aut}(\Psi^{\mathbb{Z}}(Z; V))$, which clearly contains $\mathrm{Aut}(V)$, the group of all automorphisms of the vector bundle V. The first goal is to identify this group, $\mathrm{Aut}(\Psi^{\mathbb{Z}}(Z; V))$.

Filtered algebra bundles $\Psi^{\mathbb{Z}} \to X$ with typical fibre $\Psi^{\mathbb{Z}}(Z; V)$ and $\mathrm{Diff}(Z)$ as structure group, recover fibre bundles with typical fibre Z and the setting of the Atiyah-Singer index theorem for families. In this case, $0 = \delta(\Psi^{\mathbb{Z}}) \in H^3(X; \mathbb{Z})$.

However it turns out that $\mathrm{Aut}(\Psi^{\mathbb{Z}}(Z; V))$ is much larger than $\mathrm{Diff}(Z)$, and our perspective leads naturally to a more general "families" index theorem, as in general, there is **no** finite dimensional fibre bundle $Z \to Y \overset{\phi}{\to} X$ as in case of the Atiyah-Singer index theorem for families, and in general

$$0 \neq \delta(\Psi^{\mathbb{Z}}) \in H^3(X; \mathbb{Z}).$$

Fourier Integral operators (FIOs)

Generating functions for canonical transformations:

Let $S : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a smooth function in a neighbourhood of $(x_0, \xi_0) \in \Omega \times \mathbb{R}^n$ such that $\frac{\partial^2 S(x,\xi)}{\partial x \partial \xi} \neq 0$.
Then $\chi(y, \xi) = (x, \eta)$ defines a **canonical transformation**, where $y = \frac{\partial S(x,\xi)}{\partial \xi}$ and $\eta = \frac{\partial S(x,\xi)}{\partial x}$. That is

$$\chi^*(\omega) = \omega$$

Here $\omega = \sum dp_i \wedge dq_i$ is the canonical symplectic form on $T^*\mathbb{R}^n$. S is called the **generating function** for χ. The converse is also true, that every canonical transformation χ has a local generating function S.
S is homogeneous of degree 1 in ξ iff Φ is homogeneous in ξ.
Ex: $S(x, \xi) = x.\xi$. Then $\eta = \frac{\partial S(x,\xi)}{\partial x} = \xi, \quad y = \frac{\partial S(x,\xi)}{\partial \xi} = x$ therefore $\Phi = I$.

Fourier Integral operators (FIOs)

Let $S(x,\xi)$ be a generating function as before, and $p(x,\xi)$ a symbol of order m. Then the associated Fourier Integral operator F of order m is

$$Fu(x) = (2\pi)^{-n} \int\int e^{i(S(x,\xi)-y.\xi)} p(x,\xi)u(y)dyd\xi.$$

and even more generally

$$(2\pi)^{-n} \int\int e^{i(\phi(x,y,\xi)} p(x,\xi)u(y)dyd\xi.$$

is a Fourier integral operator, where ϕ is a nondegenerate phase function, homogeneous of degree 1 in ξ.

Ex: If $S(x,\xi) = x.\xi$, then we recover pseudodifferential operators, i.e. $\Psi^\bullet(Z) \subset \mathcal{F}^\bullet(\chi)$

FIOs

Let $\chi : S^*Z \longrightarrow S^*Z$ be a canonical transformation between two compact manifolds, ie a contact diffeomorphism between their cosphere bundles.

Let $\mathcal{F}^s(\chi)$ denote the linear space of Fourier integral operators (FIOs) associated to χ of complex order s.
(Schwartz kernels known as Lagrangian distributions wrt the conic Lagrangian manifold associated to the graph of χ)

So $F \in \mathcal{F}^s(\chi)$ is a linear operator $F : \mathcal{C}^\infty(Z; V) \longrightarrow \mathcal{C}^\infty(Z; V)$ where V is a vector bundle over Z.

FIOs & automorphisms of pseudodifferential operators

One of our main results is an extension of the main theorem by Duistermaat-Singer [DS] characterizing the automorphisms of pseudodifferntial operators.

Theorem (V.M.-R.B. Melrose)

For a compact manifold Z and a vector bundle V over Z, every linear order-preserving algebra isomorphism $\Psi^{\mathbb{Z}}(Z;V) \longrightarrow \Psi^{\mathbb{Z}}(Z;V)$ is of the form

$$\Psi^{\mathbb{Z}}(Z;V) \ni A \longrightarrow FAF^{-1} \in \Psi^{\mathbb{Z}}(Z;V) \qquad (1)$$

where $F \in \mathcal{F}^{s}(\chi)$ for some canonical isomorphism χ and some complex order $s \in \mathbb{C}$, has inverse $F^{-1} \in \mathcal{F}^{-s}(\chi^{-1})$ and is determined up to a non-vanishing multiplicative constant by the algebra isomorphism.

Remarks/summary

- In other words,

$$\mathrm{Aut}(\Psi^{\mathbb{Z}}(Z;V)) = \mathrm{PGL}(\mathcal{F}^{\bullet}(Z;V))$$

- We replace arguments of [DS] by microlocal ones, which removes the assumption $H^1(S^*Z) = 0$ made by [DS], and which also enables us to extend it to the case with coefficients in a vector bundle, generalising [DS].

- There are plenty of outer automorphisms of $\Psi^{\mathbb{Z}}(Z;V)$, since the inner automorphisms are $\mathrm{PGL}(\Psi^{\bullet}(Z;V))$, which is a much smaller subgroup of $\mathrm{PGL}(\mathcal{F}^{\bullet}(Z;V))$.
Roughly speaking, the difference is the group of contact diffeomorphisms of S^*Z.

Construction of pseudodifferential algebra bundles

Consider a principal bundles $\mathcal{F} \to X$ over X with structure group $\mathsf{PGL}(\mathcal{F}^\bullet(Z; V))$.
Then we can form the associated algebra bundle of pseudodifferential operators

$$\Psi^{\mathbb{Z}} = \mathcal{F} \times_{\mathsf{PGL}(\mathcal{F}^\bullet(Z;V))} \Psi^{\mathbb{Z}}(Z, V).$$

And conversely, every algebra bundle of pseudodifferential operators is of this form, since we have shown that $\mathrm{Aut}(\Psi^{\mathbb{Z}}(Z; V)) = \mathsf{PGL}(\mathcal{F}^\bullet(Z; V))$.

It remains to show that there are examples of purely infinite dimensional pseudodifferential algebra bundles.

Construction of purely infinite dimensional pseudodifferential algebra bundles

Theorem

Whereas there are no non-trivial fibre bundles over $S^n,\ n \geq 2$ with typical fibre $\Sigma_g,\ g \geq 2$, there are infinitely many topologically distinct principal $\mathrm{GL}(\mathcal{F}^0(\Sigma_g))$ bundles over $S^n,\ n \geq 2$. Similarly there are infinitely many topologically distinct principal $\mathrm{PGL}(\mathcal{F}^0(\Sigma_g))$ bundles over $S^n,\ n \geq 2$.

These principal $\mathrm{GL}(\mathcal{F}^0(\Sigma_g))$ and $\mathrm{PGL}(\mathcal{F}^0(\Sigma_g))$ bundles over $S^n,\ n \geq 2$ are all purely infinite dimensional, and not arising from any fibre bundles over $S^n,\ n \geq 2$ with typical fibre $\Sigma_g,\ g \geq 2$.

Construction of purely infinite dimensional pseudodifferential algebra bundles

Key computation:

$$\pi_k(\mathrm{PGL}(\mathcal{F}^0(\Sigma_g))) \cong \begin{cases} \mathbb{Z}^{2g} & \text{if } k > 2 \text{ is even;} \\ \mathbb{Z}^{2g+1} \oplus \mathbb{Z}_{2-2g} & \text{if } k > 1 \text{ is odd} \end{cases}$$

Also

$$\mathbb{Z}^{2g} \hookrightarrow \pi_2(\mathrm{PGL}(\mathcal{F}^0(\Sigma_g)))$$

and

$$\mathbb{Z}^{2g+2} \oplus \mathbb{Z}_{2-2g} \twoheadrightarrow \pi_1(\mathrm{PGL}(\mathcal{F}^0(\Sigma_g))).$$

So there are plenty of topologically nontrivial principal $\mathrm{PGL}(\mathcal{F}^0(\Sigma_g)$-bundles over spheres.

Construction of purely infinite dimensional pseudodifferential algebra bundles

In contrast, one knows by a classical result of C. Earle, J. Eells,

$$\pi_k(\mathrm{Diff}(\Sigma_g)) = 0, \qquad \textit{if } k > 0.$$

This shows in particular that there are no nontrivial fibre bundles

$$\begin{tikzcd} \Sigma_g \arrow[r] & Y \arrow[d, "\phi"] \\ & S^{k+1} \end{tikzcd}$$

where $k > 0$.

Geometry of the central extension

We now want to study the geometry of the central extension

$$\mathbb{C}^* \longrightarrow \mathrm{GL}(\mathcal{F}^\bullet(Z)) \longrightarrow \mathrm{PGL}(\mathcal{F}^\bullet(Z)) = \mathrm{Aut}(\Psi^{\mathbb{Z}}(Z)).$$

Consider the **Cartan-Maurer 1-form** Θ on $\mathrm{GL}(\mathcal{F}^\bullet(Z))$, which is a Lie algebra valued differential 1-form on $\mathrm{GL}(\mathcal{F}^\bullet(Z))$,

$$\Theta \in \Omega^1(\mathrm{GL}(\mathcal{F}^\bullet(Z))) \otimes \Psi^\bullet(Z),$$

where $\Psi^\bullet(Z)$ is identified with the Lie algebra of $\mathrm{GL}(\mathcal{F}^\bullet(Z))$, satisfying the following two properties:

1. Θ is $\mathrm{GL}(\mathcal{F}^\bullet(Z))$-invariant under the left action of $\mathrm{GL}(\mathcal{F}^\bullet(Z))$ on itself and the induced infinitesmal action on the Lie algebra $\Psi^\bullet(Z)$;
2. The contraction $\iota_V(\Theta) = V$ for all $V \in \Psi^\bullet(Z)$.

Informally, $\Theta_F = F^{-1}dF$ for $F \in \mathrm{GL}(\mathcal{F}^\bullet(Z))$.

Traces used in the construction

Let Q be a positive elliptic differential operator on Z of order one, and let

$$\mathrm{Tr}_Q : \Psi^{\mathbb{Z}}(Z) \longrightarrow \mathbb{C},$$

denote the **regularized trace** with respect to Q, defined as

$$\begin{aligned}\mathrm{Tr}_Q(A) &= \text{finite part}\Big|_{z=0}\ \mathrm{Tr}(Q^{-z}A)\\ &= \lim_{z\to 0}\left(\mathrm{Tr}(Q^{-z}A) - \frac{1}{z}\,\mathrm{Tr}_R(A)\right)\end{aligned}$$

where $\mathrm{Tr}_R(A)$ is the **residue trace**, defined as

$$\mathrm{Tr}_R(A) = \lim_{z\to 0} z\,\mathrm{Tr}(Q^{-z}A)$$

It is also determined by the symbol of A,

$$\mathrm{Tr}_R(A) = \int_{S^*Z} \mathrm{tr}(\sigma_{-n}(A)(x,\xi))dxd\xi$$

Traces used in the construction

where $n = \dim(Z)$. What is unusual about this definition is that the order $(-n)$ symbol isn't invariantly defined, so it depends on the local chart, but yet the integral determining the trace is well defined.

The regularized trace Tr_Q extends the operator trace on the ideal of trace class operators on $L^2(Z)$, although Tr_Q is itself **not** a trace.

The residue trace Tr_R is a trace, but vanishes on trace class pseudodifferential operators.

The **trace defect formula** relates the two traces;

$$\mathrm{Tr}_Q([A, B]) = \mathrm{Tr}_R(\delta_Q(A)B)$$

where $\delta_Q = [\log(Q), .]$ is the outer derivation determined by Q.

Geometry of the central extension

We can extend Tr_Q to be $\Omega^1(\mathrm{GL}(\mathcal{F}^\bullet(Z)))$-linear, and we denote the extension by the same symbol, that is, we denote $1 \otimes \mathrm{Tr}_Q$ by Tr_Q. Then

$$\mathrm{Tr}_Q : \Omega^1(\mathrm{GL}(\mathcal{F}^\bullet(Z))) \otimes \Psi^\bullet(Z) \longrightarrow \Omega^1(\mathrm{GL}(\mathcal{F}^\bullet(Z))).$$

We can similarly extend Tr_R to be $\Omega^*(\mathrm{GL}(\mathcal{F}^\bullet(Z)))$-linear, and we denote the extension by the same symbol, that is, we denote $1 \otimes \mathrm{Tr}_R$ by Tr_R.

Geometry of the central extension

Theorem

Let $Q \in \Psi^1(Z)$ be a positive elliptic differential operator on Z, such that the regularized trace is normalized, $\mathrm{Tr}_Q(\mathrm{I}) = 1$. *Then the regularized trace of the Cartan-Maurer* Θ *on* $\mathrm{GL}(\mathcal{F}^\bullet(Z))$,

$$A_Q = \mathrm{Tr}_Q(\Theta),$$

is a **connection 1-form** *on the previous central extension. Let $Q_1 \in \Psi^1(Z)$ be another positive elliptic differential operator on Z, such that* $\mathrm{Tr}_{Q_1}(\mathrm{I}) = 1$. *Then*

$$A_Q - A_{Q_1} = -\,\mathrm{Tr}_R(\Theta(\log(Q) - \log(Q_1))), \qquad (2)$$

where Tr_R *denotes the residue trace,* $\log(Q) - \log(Q_1) \in \Psi^0(Z)$ *and the RHS of (2) is a basic 1-form on* $\mathrm{GL}(\mathcal{F}^\bullet(Z))$.

Geometry of the central extension

Lemma

In the notation above, the curvature Ω_Q of the connection $\mathrm{Tr}_Q(\Theta)$ on the relevant central extension is is

$$\Omega_Q(\psi_1, \psi_2) = \mathrm{Tr}_R(\delta_Q(\psi_1)\psi_2), \qquad \forall\, \psi_1, \psi_1 \in \Psi^{\mathbb{Z}}(Z),$$

where Tr_R denotes the Guillemin-Wodzicki trace and δ_Q is the derivation $[\log(Q), \cdot]$. That is,

$$\Omega_Q = \mathrm{Tr}_R(\delta_Q(\Theta) \wedge \Theta).$$

Moreover the transgression formula is

$$\Omega_Q - \Omega_{Q_1} = -d\, \mathrm{Tr}_R\left(\Theta(\log(Q) - \log(Q_1))\right).$$

FIO bundle

We have the lifting bundle gerbe,

$$
\begin{tikzcd}
\mathbb{C}^* \arrow[d] & \\
\mathrm{GL}(\mathcal{F}^\bullet(Z)) \arrow[d] & \\
\mathrm{PGL}(\mathcal{F}^\bullet(Z)) \arrow[r] & \mathbf{F} \arrow[d, "\pi"] \\
& X
\end{tikzcd}
$$

where

$$\mathbf{F} = \mathrm{Aut}(\Psi^{\mathbb{Z}}).$$

A connection for the projective FIO bundle

Since we have the curvature differential form Ω_Q for the central extension we can consider the problem of lifting the $\mathrm{PGL}(\mathcal{F}^\bullet(Z))$-bundle $\mathbf{F} \to X$ to a $\mathrm{GL}(\mathcal{F}^\bullet(Z))$-bundle $\widehat{\mathbf{F}} \to X$. The obstruction to doing this is the Dixmier-Douady class, and our goal is to calculate it.

We first need to define a fibrewise regularized trace on $\Psi^{\mathbb{Z}}$. To do this, choose a holomorphic family $Q(z)$ that is a section of the projective bundle of pseudodifferential algebra bundle $\Psi^{\mathbb{Z}}$. Such a holomorphic family can be explicitly constructed using a partition of unity.

A connection for the projective FIO bundle

A **Higgs field** is a map $\Phi\colon \mathbf{F} \to \Psi^{\mathbb{Z}}(Z;V)$ satisfying

$$\Phi(p\gamma) = ad(\gamma^{-1})\Phi(p) + \gamma^{-1}\partial\gamma \tag{3}$$

for all $\gamma \in \mathrm{PGL}(\mathcal{F}^{\bullet}(Z;V))$.

It is clear that Higgs fields exist, since they exist when **F** is trivial and convex combinations of Higgs fields are also Higgs fields, we can use a partition of unity to construct a Higgs field in general.

Let Φ' be another Higgs field. Then clearly

$$(\Phi - \Phi')(p\gamma) = ad(\gamma^{-1})(\Phi - \Phi')(p) \tag{4}$$

so that $(\Phi - \Phi')$ is a section of the adjoint bundle.

A connection for the projective FIO bundle

A **curving or B-field** is given by

$$B = B_{Q,\Phi} = \frac{i}{2\pi} \operatorname{Tr}_R(\tfrac{1}{2} A \wedge \delta_Q A - F \wedge \Phi). \tag{5}$$

It follows immediately that

$$B_{Q,\Phi} - B_{Q',\Phi} = \frac{i}{2\pi} \operatorname{Tr}_R(\tfrac{1}{2} A \wedge [P, A] - F \wedge \Phi).$$

A connection for the projective FIO bundle

Theorem

Let $\mathbf{F} \to X$ *be a principal* $\mathrm{PGL}(\mathcal{F}^\bullet(Z))$*-bundle. Let* A *be a connection on* $\mathbf{F}$ *with curvature* F *and let* Φ *be a Higgs field for* $\mathbf{F}$, *and* $\nabla^Q\Phi = d\Phi + [A, \Phi] - \delta_Q A$. *Then a B-field is given by equation (4) and the Dixmier-Douady class of* $\mathbf{F}$ *is represented in de Rham cohomology by the closed 3-form on* X*,*

$$H_Q = -\frac{i}{2\pi}\,\mathrm{Tr}_R(F \wedge \nabla^Q\Phi).$$

The transgression formula for H_Q *as* Q *varies is given by,*

$$H_Q - H_{Q_1} = -\frac{i}{2\pi} d\,\mathrm{Tr}_R(A \wedge [P, A]), \tag{6}$$

where $P = \log(Q) - \log(Q_1)$ *is an order zero, pseudodifferential section of the pseudodifferential algebra bundle* $\Psi^{\mathbb{Z}}$ *over* X*.*