Honeycomb Toroidal Graphs

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These are maps on the torus for which every vertex has valency 3 and the faces are all hexagons.

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By far the best known and most widely studied family of vertex-transitive graphs are Cayley graphs. They are defined as follows.

Cayley Graphs

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Define the Cayley graph on G with connection set S as follows. The vertices are the elements of G and for each $g \in G$ there are edges to all vertices of the form gs as s runs through S. The notation is $\mathrm{Cay}(G; S)$.

Cayley Graphs

Note that left multiplication by an element x of G is an automorphism of $\mathrm{Cay}(G; S)$ because the edge $[g, gs]$ is mapped to the edge $[xs, xgs]$. Thus, $\mathrm{Aut}(\mathrm{Cay}(G; S))$ contains the left-regular representation of G.

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This suggests the classical theorem of Sabidussi (1958).

Theorem. A graph X is a Cayley graph if and only if $\mathrm{Aut}(X)$ contains a regular subgroup.

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An important feature of a regular group G is that for each x, y in the set of objects being permuted, there exists a unique $f \in G$ such that $f(x) = y$. Thus, arbitrarily label a vertex u of the graph with $1 \in G$. Now label any other vertex v with the unique element of G that maps u to v. It is not hard to show that this labelling with elements of G satisfies the definitions of a Cayley graph.

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The first thread is that we were told that our result settled the unsolved problem raised by Altshuler mentioned at the beginning. We shall see later that, in fact, this is not true.

The Rise Of A Project

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A graph X is Hamilton-connected if for every pair of vertices u and v, there is a Hamilton path whose terminal vertices are u and v. Similarly, a bipartite graph is Hamilton-lacable if for every pair of vertices in opposite parts, there is a Hamilton path joining them.

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My project is to try to show that Cayley graphs on dihedral groups are either Hamilton-connected or Hamilton-laceable depending on whether or not they are bipartite. This is a much stronger condition but it does allow induction to be used.

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If we can prove that a connected Cayley digraph on a dihedral group whose connection set consists of three reflections is Hamilton-lacebale, then my project can be carried out to completion. So what are these roadblock graphs like?

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Given that the dihedral group is generated by ρ and τ, where $|\rho| = n$, $|\tau| = 2$ and $\tau\rho\tau = \rho^{-1}$, we may assume the connection set for our Cayley graph is $\{\tau, \rho^i\tau, \rho^j\tau\}$.

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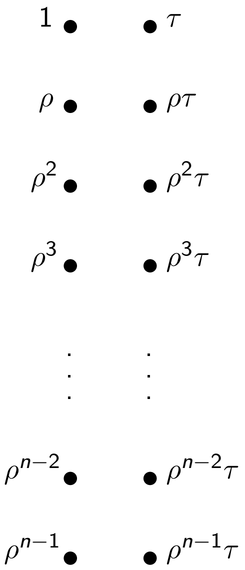
Let's take a look at what is happening. Start by considering the subgraph generated by τ and $\rho^i\tau$.

The Roadblock



Display the vertices of the Cayley graph as shown on the left. The left column has the vertices of the subgroup $\langle\rho\rangle$ and the right column has the vertices of the right coset $\langle\rho\rangle\tau$.

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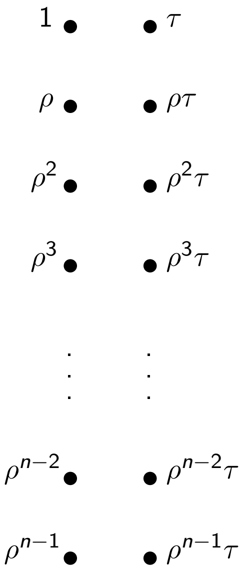

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Let's now examine the subgraph given by the elements τ and $\rho^i\tau$.

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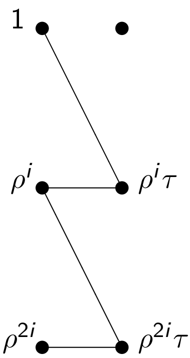

The beginning of a cycle C generated by τ and $\rho^i\tau$ tells us exactly what is happening. The cosets are cyclically labelled and we are jumping by i each time return to a coset.

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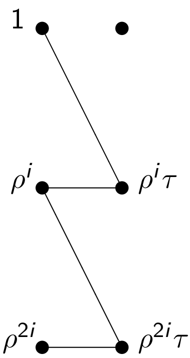

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Thus, there are $\gcd(n, i)$ cycles comprising the graph so each has length $\frac{2n}{\gcd(n,i)}$.

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The subgraph generated by $\rho^i\tau$ and τ is connected if and only if $\gcd(n, i) = 1$. So when there are two or more cycles, the third element $\rho^j\tau$ must connect the cycles together if the graph is to be connected.

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The element $\rho^j\tau$ generates an edge joining 1 in C and $\rho^j\tau$ in another cycle C'. Note that $\rho^j\tau$ generates an edge from ρ^i to $\rho^{j+i}\tau$, but this vertex also lies in C'. So we see that the edges generated by $\rho^j\tau$ join the vertices of C that lie in $\langle\rho\rangle$ to the vertices of C' that lie in $\langle\rho\rangle\tau$.

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Let's now draw this type of graph in a different manner which then yields a "familiar" family.

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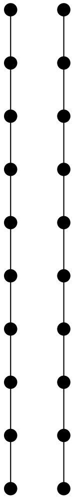

Draw two cycles vertically that are joined by $\rho^j\tau$-edges.

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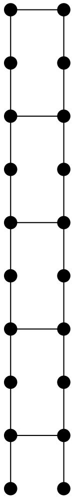

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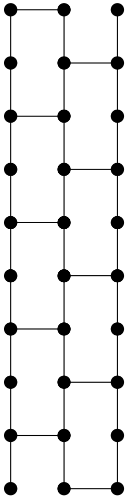

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If there is another cycle, then it looks like this.

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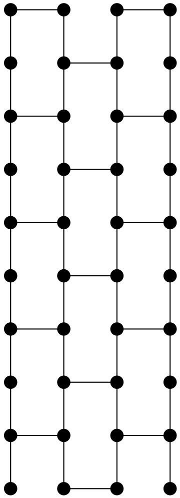

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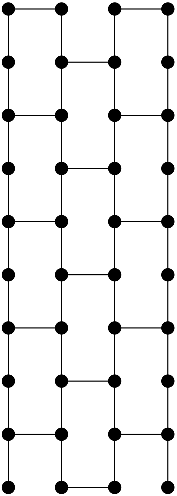

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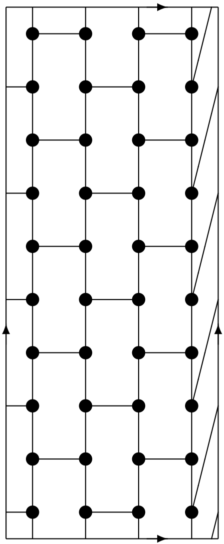

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Here the jump is 2 and we have an embedding on the torus. It is a regular (6,3) map.

Honeycomb Toroidal Graphs

This is the family graphs we are considering and the preceding description is how we view them. They are called honeycomb toroidal graphs and are described as $\mathrm{HTG}(m, n, \ell)$, where m is the number of vertical cycles, n is the length of the cycles so that $n \geq 4$ and is even, and ℓ is the jump from the last cycle to the first.

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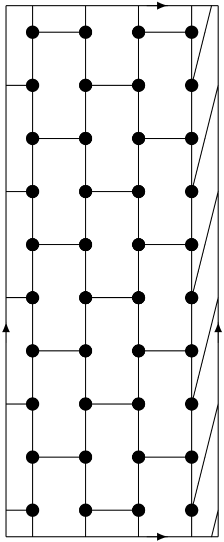

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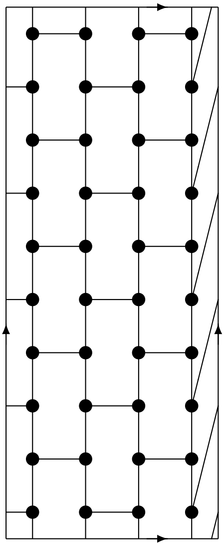

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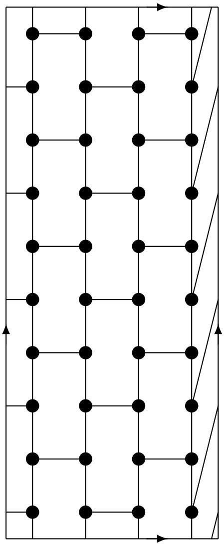

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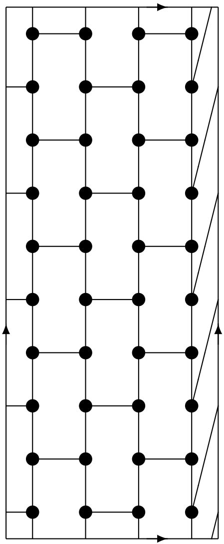

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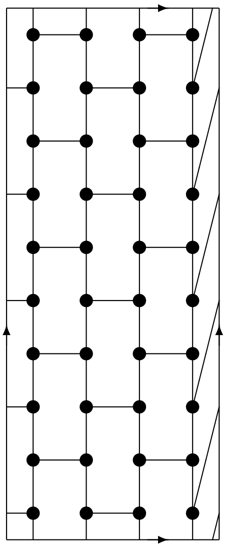

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Note that $fg = gf$ so that $\langle f, g\rangle$ is an abelian group with two orbits. Because the restriction of $\langle f, g\rangle$ to an orbit is faithful, $|\langle f, g\rangle| = \frac{mn}{2}$.

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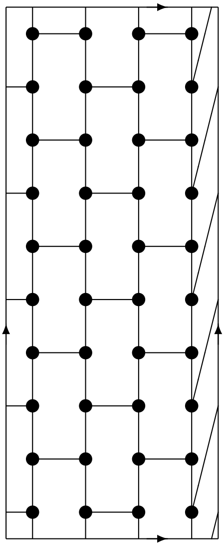

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There is an involution τ interchanging the two orbits and satisfying $\tau f \tau = f^{-1}$ and $\tau g \tau = g^{-1}$. Thus, $\langle f, g, \tau\rangle$ is a regular group.

Honeycomb Toroidal Graphs

Theorem. (Alspach and Dean, 2009). Honeycomb toroidal graphs are Cayley graphs on generalized dihedral groups.

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Now let's look at the hamiltonicity project. Use $\mathrm{HTG}(m, 24, 6)$ as an example.

Hamiltonicity Project

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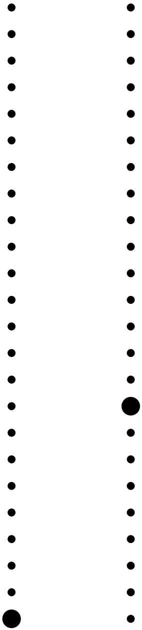

Suppose we want a Hamilton path joining the two vertices shown.

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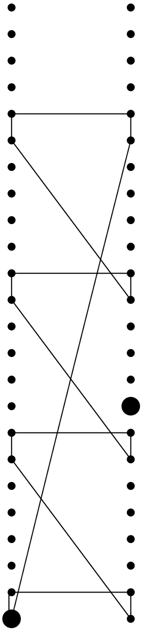

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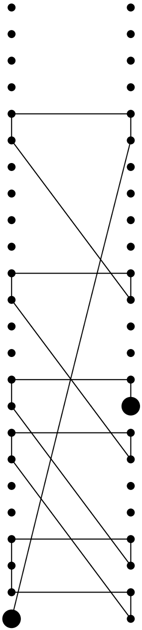

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Remove the edge shown from the cycle and move up to the next cycle. Here we end up with a path to the target vertex but it is not a Hamilton path.

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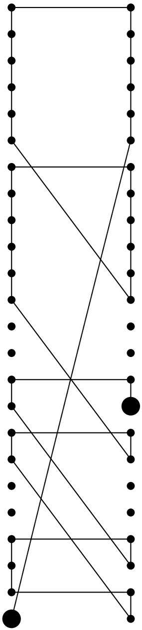

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Alter the path along the top two horizontal edges as shown. Do the same for the other two gaps to obtain a desired Hamilton path.

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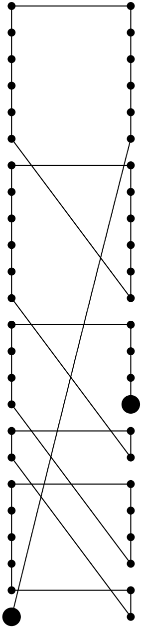

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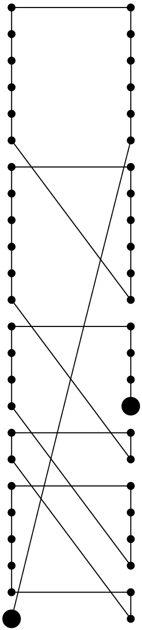

Suppose you want to add two columns between the the first and second column.

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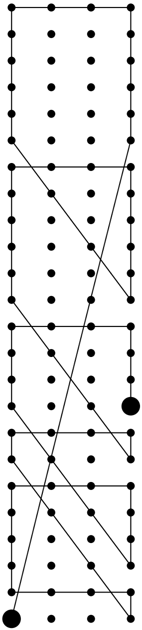

Suppose you want to add two columns between the the first and second column.

Change the horizontal 3-paths as shown next.

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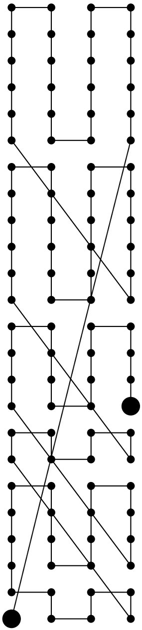
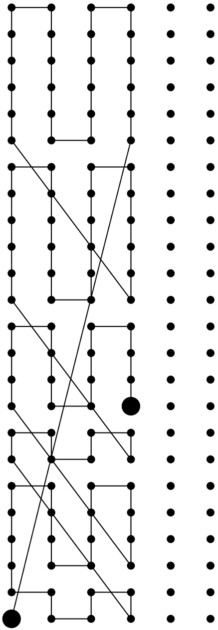


Suppose you want to add two columns between the the first and second column.

Change the horizontal 3-paths as shown next.

We now have a Hamilton path joining them.

Hamiltonicity Project



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Suppose we now wish to add two columns to the right of the target vertex.

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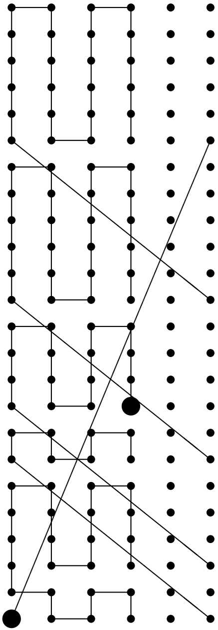

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Detach and reattach the jump edges as shown.

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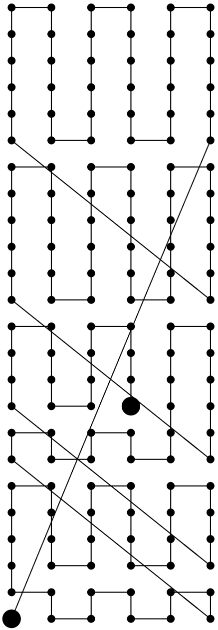

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Fill in to capture all the vertices as shown.

Hamiltonicity Project

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Theorem (Alspach, Chen and Dean, 2010) The honeycomb toroidal graph $\mathrm{HTG}(m, n, \ell)$ is Hamilton-laceable whenever m is even.

Corollary. If X is a connected Cayley graph of valency at least 3 on a generalized dihedral group whose order is divisible by 4, then X is Hamilton-laceable if it is bipartite or Hamilton-connected if it is not bipartite.

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The situation for m odd is still up in the air although if the analogous result is true for $m = 1$, then it is true for all odd m which would, of course, settle the problem.

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Sean McGuinness has shown that if $n \geq 4\ell^2$, then $\mathrm{HTG}(m, n, \ell)$ is Hamilton-laceable (unpublished).

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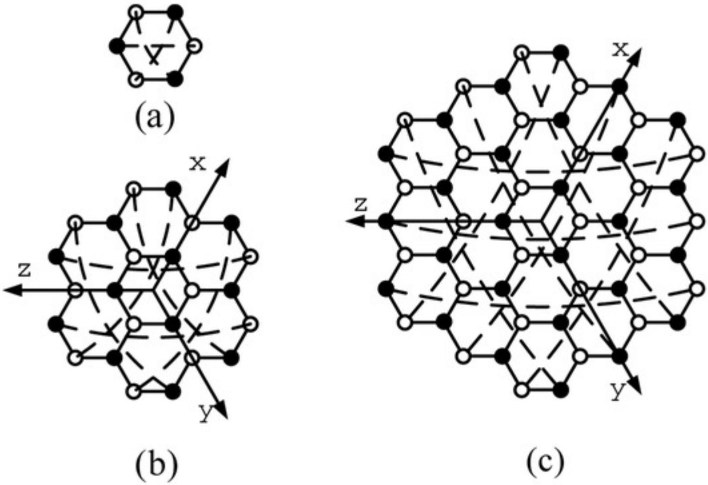
A nice way to describe this special family is to start with a Hamilton cycle of even length n. Then choose an odd integer $\ell \leq n/2$ and join every even labelled vertex (assuming they are labelled cyclically using $0, 1, \ldots, n-1$) i to $i + \ell$.

Computer Science Viewpoint



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The rhombic honeycomb torus is $\mathrm{HTG}(m, 2m, m)$, $m \geq 2$.

Connectivity

Every vertex-transitive graph has edge connectivity equal to its valency. Thus, $\mathrm{HTG}(m, n, \ell)$ is 3-edge-connected.

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The importance of the preceding results is that between any two distinct vertices of $\mathrm{HTG}(m, n, \ell)$ there are three internally disjoint paths joining them and there are three edge-disjoint paths joining them. This is of interest because of how many faults the network can tolerate.

Cycle Spectrum

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Proposition (Alspach and Connor, 2017). The honeycomb toroidal graph $\mathrm{HTG}(m, n, \ell)$ has girth 4 if and only if it satisfies one of the following conditions:

- $n = 4$,
- $m = 1$ and $\ell \in \{3, n-3\}$,
- $m = 1, n \equiv 0 (\mathrm{mod}\ 4)$ and $\ell \in \{n/2 - 1, n/2 + 1\}$,
- $m = 1, n \equiv 2 (\mathrm{mod}\ 4)$ and $\ell = n/2$, and
- $m = 2$ and $\ell \in \{0, 2, n-2\}$.

Otherwise, it has girth 6.

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A connected bipartite Cayley graph of valency at least 3 on an abelian group is even pancyclic so that we expect honeycomb toroidal graphs to be rich in even length cycles as the underlying group is “close” to being abelian.

I had Josh Connor look at this problem for his AMSI Summer Research Project over the 2017–2018 break.

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Theorem (Connor, 2018). The honeycomb toroidal graph $\mathrm{HTG}(m, n, \ell)$ has cycles of all lengths r, where $r \equiv 2 (\mathrm{mod}\ 4)$ and $6 \leq r \leq mn$.

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Research Problem: Determine the cycle lengths occurring in honeycomb toroidal graphs.

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Diameter is a graph parameter of interest because it gives a lower bound on the time required to propogate a message to all the vertices of a graph. This parameter has been studied but primarily for the particular honeycomb toroidal graphs that have been consider by researchers in computer architecture.

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Theorem (Yang, 2004). The diameter of $\mathrm{HTG}(m, 2m, m)$ is

- $\lfloor \frac{4m}{3} \rfloor$ when $m \equiv 1, 4 (\mathrm{mod}\ 6)$, and
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Theorem (Stojmenovic, 1997). The diameter of $\mathrm{HTG}(m, 6m, 3m)$ is $2m$.

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Research Problem: Determine the diameter of an arbitrary $\mathrm{HTG}(m, n, \ell)$.

Path Lengths

If the distance between two vertices $u, v \in \mathrm{HTG}(m, n, \ell)$ is L, then it's possible there could be paths of all lengths $L, L+2, \ldots,$ through mn or $mn-1$ depending on the parity of L.

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This problem has been studied but in a very limited way. That suggests

Research Problem: Study the path length spectrum problem for honeycomb toroidal graphs.

Other Parameters

There are other parameters associated with these highly symmetric graphs of small valency in which the computer scientists working in this area have an interest. They either have been investigated almost not at all or just for the special honeycomb toroidal graphs mentioned earlier.

Thank You