ORIENTATION OF PIECEWISE POWERS OF A MINIMAL HOMEOMORPHISM

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ABSTRACT. We show that given a compact minimal system (X, g), and given an element h of the topological full group $\tau[g]$ of g, then the infinite orbits of h admit a locally constant 'orientation' with respect to the orbits of g. We use this to obtain a clopen partition of (X, h) into minimal and periodic parts, showing in particular that h is pointwise almost periodic. We also use the orientation of orbits to give another interpretation of the index map and to explore the role in $\tau[g]$ of the submonoid generated by the induced transformations of g. Finally, we consider the problem, given a homeomorphism h of the Cantor set X, of determining whether or not there exists a minimal homeomorphism g of X such that $h \in \tau[g]$.

1. INTRODUCTION

Let X be a Hausdorff topological space and let $g \in \mathrm{Homeo}(X)$; write $\langle g \rangle := \{g^n \mid n \in \mathbf{Z}\}$ for the group of homeomorphisms generated by g. We say $g \in \mathrm{Homeo}(X)$ is **minimal** if X is nonempty, and whenever K is a proper closed subspace of X such that $gK \subseteq K$, then $K = \emptyset$. The **topological full group** $\tau[g]$ of g consists of all homeomorphisms h of X such that for every $x \in X$, there is a neighbourhood U of x and $n \in \mathbf{Z}$ such that $h(y) = g^n(y)$ for all $y \in U$. In other words, $\tau[g]$ consists of those homeomorphisms that are 'piecewise' in $\langle g \rangle$.

The topological full group was introduced in the 1990s as a tool to study a minimal homeomorphism of the Cantor set by algebraic methods. Its fundamental properties were established by Giordano, Putnam and Skau in [3], building on the C^*-algebra approach used in [2]. (See also [11].) Topological full groups also turn out to have remarkable properties from a group-theoretic perspective, providing for instance the first known examples of infinite finitely generated simple amenable groups (see [4]). Their theory has been developed and generalized by many different authors, for example to the setting of étale groupoids; see [8] for a survey of some recent developments.

The definition of the topological full group makes sense for any homeomorphism (or indeed for much more general classes of action or partial action). However, in the case that (X, g) is a **compact minimal system**, that is, X is an infinite compact Hausdorff space and g is a minimal homeomorphism of X, the group $\tau[g]$ has a special structure arising from the partial order that the action of g induces on the space. In this article we identify two types of 'positive' element of $\tau[g]$ and derive consequences for the structure of general elements of $\tau[g]$.

Definition 1.1. Given a compact minimal system (X, g), define a partial order $\leq_g$ on X by setting $x \leq_g y$ if $y = g^t x$ for some $t \geq 0$. Given $h \in \tau[g]$ and a $\langle h \rangle$-orbit Y, we say Y is **positive** (with respect to g) if for all $y, z \in Y$, there is $n \in \mathbf{N}$ such that $h^{n'} y \geq_g z \geq_g h^{-n'} y$ for all $n' \geq n$, and **strongly positive** (with respect to g) if for all $y \in Y$ we have $hy \geq_g y$. The orbit is **(strongly) negative** with respect to g if it is (strongly) positive with respect to g^{-1}. A **trivial orbit** is a fixed point of h; note that this is the only kind of orbit that is both positive and negative. Say that h is **(strongly) positive** if it is (strongly) positive on every $\langle h \rangle$-orbit; write $\tau_+[g]$ for the set of positive elements of $\tau[g]$ with respect to g and $\tau_>[g]$ for the set of strongly positive elements with respect to g.

Our first main result is that given a compact minimal system (X, g), every element of the topological full group can be naturally partitioned into a positive, negative and periodic part. (See also [6, Proposition 4.13].)

Theorem 1.2 (See §3)**.** *Let (X, g) be a compact minimal system and let $h \in \tau[g]$. Then X admits a partition into clopen sets*

$$X = X_p \sqcup X_+ \sqcup X_-,$$

where X_p is the union of finite orbits, X_+ is the union of nontrivial positive orbits, and X_- is the union of nontrivial negative orbits of h. As a result, every element $h \in \tau[g]$ can be written uniquely as $h_p h_+ h_-$, where h_p, h_+ and h_- are elements of $\tau[g]$ with disjoint clopen support, we have $h_+, h_-^{-1} \in \tau_+[g]$, and there is some $n > 0$ such that $h_p^n = \mathrm{id}_X$.

We refer to the partition of $h \in \tau[g]$ given by Theorem 1.2 as the **sign partition** of h (with respect to g). Closely related to the existence of the sign partition of h is another kind of partition that represents an intrinsic property of h (that is, without any direct reference to g), and is analogous to a phenomenon observed by M. Keane ([5]) in the context of interval exchange transformations.

Definition 1.3. Let h be a homeomorphism of an infinite compact Hausdorff space X. We say h admits a **minimal-periodic partition** if there is a partition of X into clopen $\langle h \rangle$-invariant spaces

$$X = \bigsqcup_{n \in \mathbf{N}} X_p(n) \sqcup \bigsqcup_{i=1}^{m} X_a(i),$$

where every $\langle h \rangle$-orbit on $X_p(n)$ has exactly n points, and h acts freely and minimally on each of the sets $X_a(1), \dots, X_a(m)$. If h admits a minimal-periodic partition, write $m(h)$ for the number m, that is, the number of infinite orbit closures of h; $X_p = \bigsqcup_{n \in \mathbf{N}} X_p(n)$; and $X_a := \bigsqcup_{i=1}^{m} X_a(i)$.

Theorem 1.4 (See §3)**.** *Let X be an infinite compact Hausdorff space and let $g \in \mathrm{Homeo}(X)$. If g admits a minimal-periodic partition, then so does every $h \in \tau[g]$.*

In particular, any compact minimal system (X, g) clearly admits a minimal-periodic partition, with empty periodic part and $m(g) = 1$, so Theorem 1.4 says in this case that every $h \in \tau[g]$ admits a minimal-periodic partition. In this case, the minimal-periodic partition of h refines the sign partition of h: each of the sets $X_p(n)$ is contained in the periodic part X_p, and each of the free minimal parts $X_a(i)$ of h is contained in either X_+ or X_-. It also follows that if g is a pointwise almost periodic homeomorphism of a locally compact Hausdorff space X, then so is every $h \in \tau[g]$ (see Corollary 3.5).

A well-known feature of the group $\tau[g]$ is the index map, which is the unique group homomorphism $I : \tau[g] \to \mathbf{Z}$ such that $I(g) = 1$. This map was introduced in [3] in the metrizable case, using an integral formula. In this article we give another construction of the index map, by counting infinite orbits up to orientation; like the integral formula, it can also be understood as an average of cocycle values (without invoking a measure on the space).

Theorem 1.5 (See Proposition 5.5 and Theorem 5.6)**.** *Let (X, g) be a compact minimal system and let $h \in \tau[g]$. Then there are nonnegative integers $o^+(h)$ and $o^-(h)$, such that every orbit of $\langle g \rangle$ contains exactly $o^+(h)$ nontrivial positive orbits of $\langle h \rangle$ and $o^-(h)$ nontrivial negative orbits of $\langle h \rangle$. Moreover, the index map is given by*

$$I(h) = o^+(h) - o^-(h) = \lim_{l \to \infty} \frac{1}{l} \sum_{1 \leq j \leq l} c_{g,h}(g^j x) \ (\forall x \in X).$$

As the name suggests, every strongly positive element is positive, but not conversely in general (see Example 4.1). The relationship between $\tau_+[g]$ and $\tau_>[g]$ can be summarized as follows.

Theorem 1.6 (See Lemma 4.2 and Proposition 4.7)**.** *Let (X, g) be a compact minimal system.*

(i) $\tau_+[g]$ is closed under conjugation in $\tau[g]$; moreover, given $h \in \mathrm{N}_{\mathrm{Homeo}(X)}(\tau[g])$, then $h\tau_+[g]h^{-1} \in \{\tau_+[g], (\tau_+[g])^{-1}\}$.

(ii) Given $h \in \tau_+[g]$, there is a unique $\tau[h]$-conjugate h' of h such that $h' \in \tau_>[g]$.

Using the relationship between positive and strongly positive elements, we obtain several equivalent descriptions of the elements h such that $\tau[h] = \tau[g]$ (Proposition 4.10).

Given a nonempty clopen subset A of X, the **induced transformation** g_A is defined by setting, for each $x \in A$, the image $g_A x$ to be $g^t x$, where t is the least positive integer such that $g^t x \in A$. The induced transformations form an interesting generating set for topological full group and also for the monoid of strongly positive elements.

Definition 1.7. In a monoid H with identity 1, an **irreducible element** is an element $a \in H \setminus \{1\}$ such that whenever $a = bc$ for $b, c \in H$, then $\{b, c\} = \{1, a\}$.

Theorem 1.8 (See §6.2)**.** *Let (X, g) be a compact minimal system.*

(i) $\tau_>[g]$ is the submonoid of $\tau[g]$ generated by the induced transformations of g; moreover, the induced transformations are exactly the irreducible elements of $\tau_>[g]$ as a monoid.

(ii) Given $h \in \tau[g]$, there is exactly one way to write h as a product

$$h = g_{A_n} \dots g_{A_2} g_{A_1} g^r$$

such that $r \in \mathbf{Z}$ and

$$A_n \subseteq \dots \subseteq A_2 \subseteq A_1 \subsetneq X.$$

It was shown in [3] that when X is the Cantor set, then the group $\tau[g]$ is a complete invariant for the flip conjugacy class of (X, g). Analogously, when X is zero-dimensional, the monoid $\tau_>[g]$ is a complete invariant for the conjugacy class of (X, g), with a straightforward description of how the space X manifests in the monoid structure (see Proposition 6.4). The monoid $\tau_>[g]$ thus provides an alternative approach to describing the dynamical system (X, g) by algebraic means.

Periodic elements of $\tau[g]$ also have a natural decomposition into 'pure cycles' of g (see §6.1).

Say that a homeomorphism h of a compact Hausdorff space X is **piecewise a power of a minimal homeomorphism** (p.p.m.) if there exists $g \in \mathrm{Homeo}(X)$ such that g

is minimal on X and $h \in \tau[g]$. As we see from Theorem 1.4, p.p.m. homeomorphisms have a special form, namely they admit a minimal-periodic partition. Now suppose that we are given some homeomorphism $h \in \mathrm{Homeo}(X)$ that admits a minimal-periodic partition. When is h p.p.m.?

First, observe that if h is p.p.m., then so are all its induced transformations on nonempty clopen subspaces: specifically, if $h \in \tau[g]$ where g is minimal, and A is a nonempty clopen subspace, then $h_A \in \tau[g_A]$. Thus the problem can be broken down as follows:

(1) Given an aperiodic homeomorphism with a minimal-periodic partition, when is it p.p.m.?
(2) Which homeomorphisms of finite order are p.p.m.?
(3) Given p.p.m. homeomorphisms h_1 of X_1 and h_2 of X_2, where h_1 has finite order and h_2 is aperiodic, when is the disjoint union of h_1 and h_2 p.p.m.?

The second and third questions turn out to have an easy answer as long as the underlying topological space has a sufficiently rich group of homeomorphisms. Say that a topological space X is a **generalized Cantor space** if X is compact, Hausdorff, perfect, zero-dimensional, and every nonempty clopen subset of X is homeomorphic to X itself (equivalently, in the algebra $\mathcal{A}$ of clopen subsets of X, every nonzero principal ideal is isomorphic to $\mathcal{A}$ itself). The most well-known examples of such spaces are the Cantor spaces 2^κ, that is, the set of functions from a set of cardinality κ to $\{0,1\}$ equipped with the topology of pointwise convergence.

Theorem 1.9 (See Proposition 7.2)**.** *Let X be a generalized Cantor space and let $h \in \mathrm{Homeo}(X)$. Suppose that h admits a minimal-periodic partition, and let X_a be the union of the infinite orbits of h. Then (X,h) is p.p.m. if and only if either X_a is empty, or (X_a,h) is p.p.m.*

For the first question, we obtain a partial answer. Given an aperiodic p.p.m. homeomorphism $h \in \mathrm{Homeo}(X)$, and given a minimal homeomorphism g such that $h \in \tau[g]$, we can take the orbit number $o_g(h)$ as a measure of the 'efficiency' with which g witnesses that h is p.p.m. Write $o_{\min}(h)$ for the least value of $o_g(h)$, as g ranges over all minimal homeomorphisms of X such that $h \in \tau[g]$. It is clear that $o_{\min}(h)$ is at least $m(h)$, the number of infinite minimal orbit closures of h; say that h is **strongly p.p.m.** if $o_{\min}(h) = m(h)$. We can characterize the aperiodic strongly p.p.m. homeomorphisms using a notion of equivalence between the infinite minimal orbit closures. We say two compact minimal systems (X_1,h_1) and (X_2,h_2) are **(flip) Kakutani equivalent** if there are nonempty clopen subsets $Y_i \subseteq X_i$ such that the induced systems $(Y_1,(h_1)_{Y_1})$ and $(Y_2,(h_2)_{Y_2})$ are (flip) conjugate.

Theorem 1.10. *(See Proposition 7.5) Let X be an infinite compact Hausdorff space and let $h \in \mathrm{Homeo}(X)$ be aperiodic. Then the following are equivalent:*

(i) h is strongly p.p.m.;
(ii) There is a partition of X into clopen spaces $X_1,\dots,X_m$ such that $(X_1,h),\dots,(X_m,h)$ are compact minimal systems that lie in a single flip Kakutani equivalence class.

We can thus restate the p.p.m. property for aperiodic homeomorphisms as follows. Say that the tuple $((X_i,h_i))_{1 \le i \le m}$ of compact minimal systems is **Kakutani compatible** if there exists a sequence of compact minimal systems $(X_1,g_1),\dots,(X_m,g_m)$, all lying in a single Kakutani equivalence class, such that $h_i \in \tau[g_i]$.

Corollary 1.11. *Let X be an infinite compact Hausdorff space and let $h \in \mathrm{Homeo}(X)$ be aperiodic. Then the following are equivalent:*

(i) h is p.p.m.;

(ii) There is a partition of X into clopen spaces $X_1, \dots, X_m$ such that $(X_1, h), \dots, (X_m, h)$ are compact minimal systems and $((X_i, h_i))_{1 \le i \le m}$ is a Kakutani compatible tuple.

What is not clear is whether Kakutani compatibility reduces to an equivalence relation defined on pairs of spaces.

Question 1.12. Let $(X_1, h_1), \dots, (X_m, h_m)$ be compact minimal systems such that the pair $((X_i, h_i), (X_{i+1}, h_{i+1}))$ is Kakutani compatible for all $1 \le i \le m-1$. Is $((X_i, h_i))_{1 \le i \le m}$ Kakutani compatible?

Structure of the article. After a short preliminary section (§2), we establish the existence of the sign partition and the minimal-periodic partition (§3). We then establish the key properties that relate the set of positive elements to the set of strongly positive elements (§4). In §5 we establish the existence and uniqueness of the index map and the orbit numbers, along with some of their properties. The next section describes the pure cycle decomposition of periodic elements (§6.1) and a normal form for arbitrary elements of the topological full group in terms of induced transformations, with consequences for the structures of $\tau[g]$ and $\tau_>[g]$ (§6.2). The final section (§7) is dedicated to results on the problem of determining whether a given homeomorphism is piecewise a power of a minimal homeomorphism.

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2. Preliminaries

2.1. **Notation.** In this section we set some notation and recall some standard concepts that will be used throughout.

Given a function $\alpha : X \to Y$, and $x \in X$, we will simply write αx to mean $\alpha(x)$, where there is no danger of confusion. Since composition of functions is associative, we can similarly write $\alpha_n \alpha_{n-1} \dots \alpha_1 x$ to mean that the sequence $\alpha_1, \dots, \alpha_n$ of functions is applied successively to x. Given a subset K of X, we define $\alpha K := \{\alpha x \mid x \in K\}$, and given a set of S of functions defined at a point x, we define $Sx := \{sx \mid s \in S\}$.

Given a topological space X, we write $\mathcal{CO}(X)$ for the set of compact open subsets of X. Note that if X is compact Hausdorff, then $\mathcal{CO}(X)$ is just the set of clopen subsets of X. We say that a locally compact Hausdorff space X is **zero-dimensional** if $\mathcal{CO}(X)$ is a base for the topology of X.

2.2. **The cocycle.** Let X be an infinite compact Hausdorff space, let g be an aperiodic homeomorphism of X and let $h \in \tau[g]$. Then for each $x \in X$ there is exactly one $t \in \mathbf{Z}$ such that $hx = g^t x$. This defines a continuous map, the **cocycle of h with respect to g**, which is described as follows:

$$c_{g,h} : X \to \mathbf{Z}; \quad \forall x \in X : hx = g^{c_{g,h}(x)} x.$$

Since the cocycle is a continuous map from a compact space to a discrete space, it takes only finitely many values. We define $|h|_g := \max\{|c_{g,h}(x)| \mid x \in X\}$. We also define a partial order $\le_g$ on X: say $x \le_g y$ if $y = g^t x$ for some $t \ge 0$.

Given $h \in \tau[g]$, where the $\langle h \rangle$-orbit $\langle h \rangle x$ is finite, its size depends continuously on $x \in X$.

Lemma 2.1. *Let X be a compact Hausdorff space, let g be an aperiodic homeomorphism of X and let $h \in \tau[g]$. Write $X_p(n)$ for the set of points such that $|\langle h \rangle x| = n$. Then $X_p(n)$ is clopen in X for all natural numbers n.*

Proof. We can obtain $X_p(n)$ as

$$X_p(n) = \{x \in X \mid c_{g,h^n}(x) = 0, \forall 1 \leq i < n : c_{g,h^i}(x) \neq 0\}.$$

Thus the condition $x \in X_p(n)$ is defined by constraints on $c_{g,h^i}(x)$ for finitely many i; it follows that $X_p(n)$ is clopen. □

2.3. **Minimality and pointwise almost periodicity.** On a compact Hausdorff space, minimality is characterized in terms of strongly invariant sets, as follows.

Lemma 2.2. *Let X be a nonempty compact Hausdorff space and let $g \in \mathrm{Homeo}(X)$. Then the following are equivalent:*

(i) g is minimal, that is, whenever K is a proper closed subspace of X such that $gK \subseteq K$, then $K = \emptyset$;
(ii) whenever K is a proper closed subspace of X such that $gK = K$, then $K = \emptyset$.

Proof. Clearly (i) implies (ii). Conversely, suppose that (ii) holds, and let K be a proper closed subspace of X such that $gK \subseteq K$. We see that in fact $g^m K \subseteq g^n K$ whenever $m \geq n \geq 0$. Set $L = \bigcap_{n \geq 0} g^n K$. Then L is compact and $gL = \bigcap_{n \geq 1} g^n K = L$ (here we use the fact that g is injective); since $L \subseteq K \neq X$, it follows that $L = \emptyset$. By compactness we must have $g^n K = \emptyset$ for some n, so in fact $K = \emptyset$, proving (i). □

A natural generalization of minimal homeomorphisms of a compact Hausdorff space are pointwise almost periodic homeomorphisms, defined as follows.

Definition 2.3. Let g be a homeomorphism of a Hausdorff space X. Given a point x and a neighbourhood U of x, define the set of **return times** $n_g(x, U) := \{n \in \mathbf{Z} \mid g^n x \in U\}$. We say g is **almost periodic at** x if $n_g(x, U)$ is a syndetic subset of $\mathbf{Z}$, that is, there exists k such that $[t, t+k] \cap n_g(x, U)$ is nonempty for every $t \in \mathbf{Z}$, for every neighbourhood U of x. We say g is **pointwise almost periodic** (p.a.p.) if it is almost periodic at every point.

Lemma 2.4 (See for instance [10, Lemma 3.5])**.** *Let g be a homeomorphism of a locally compact Hausdorff space X. Then g is p.a.p. if and only if, for every $x \in X$, the orbit closure $\overline{\langle g \rangle x}$ is a compact minimal g-space.*

2.4. **Induced maps.** Given a p.a.p. homeomorphism h, we can use the return times to define the **induced transformation** on a clopen subspace.

Definition 2.5. Let g be a p.a.p. (for example, minimal) homeomorphism of a compact Hausdorff space X and let Y be a clopen subset of X. Then for all $x \in Y$, by Lemma 2.4 there exists $n > 0$ such that $g^n x \in Y$. Define the **induced transformation** $g_Y : Y \to Y$ by setting $g_Y x = g^t x$, where t is the least element of $n_g(x, Y) \cap \mathbf{N}_{>0}$.

Given homeomorphisms $h_i \in \mathrm{Homeo}(X_i)$, define the **join** $h_1 \sqcup h_2$ to be the homeomorphism g of $X_1 \sqcup X_2$ given by $gx = h_i x$ for all $x \in X_i$.

The p.a.p. property ensures that induced transformations are well-defined and well-behaved. We see in particular that all induced transformations of a p.a.p. homeomorphism belong to its topological full group. The following basic facts will be used repeatedly without further comment.

Lemma 2.6. *Let h be a p.a.p. homeomorphism of a compact Hausdorff space X and let Y be a nonempty clopen subset of X.*

(i) We have $(h^{-1})_Y = (h_Y)^{-1}$.
(ii) h_Y is a homeomorphism of Y and the join $h_Y \sqcup \mathrm{id}_{X \smallsetminus Y}$ belongs to $\tau[h]$.
(iii) Given $x \in Y$, then $\langle h_Y \rangle$ acts transitively on $\langle h \rangle x \cap Y$.
(iv) h_Y is p.a.p. on Y. If h is minimal, then so is h_Y.

Proof. (i) Let $x, y \in Y$ and write $(x, y) \in R$ if $y = h_Y x$. We see that the statement $(x, y) \in R$ is equivalent to each of the following:

$y = h^t x$, where t is the least positive integer such that $h^t x \in Y$;
$x = (h^{-1})^t y$, where t is the least positive integer such that $(h^{-1})^t y \in Y$.

The p.a.p. property ensures the existence of such integers t, so for all x there is a unique y such that $(x, y) \in R$ and vice versa. We conclude that h_Y is bijective and that $(h^{-1})_Y = (h_Y)^{-1}$.

(ii) We have already seen that h_Y is bijective, so the join $h_Y \sqcup \mathrm{id}_{X \smallsetminus Y}$ is bijective. Define $f : X \to \mathbf{N}$ by setting $f(x)$ to be the least positive integer such that $h^{f(x)} x \in Y$ if $x \in Y$, and $f(x) = 0$ otherwise. Then f is well-defined by the p.a.p. property. We see that f is continuous, since h is a homeomorphism and Y is clopen. Thus $h_Y \sqcup \mathrm{id}_{X \smallsetminus Y} \in \tau[h]$; in particular,

$$h_Y \sqcup \mathrm{id}_{X \smallsetminus Y} \in \mathrm{Homeo}(X) \text{ and } h_Y \in \mathrm{Homeo}(Y).$$

(iii) Clearly $\langle h_Y \rangle x$ is a subset of $\langle h \rangle x \cap Y$. From the definition, we see that $\langle h_Y \rangle x$ contains $\{h^t x \mid t \geq 0, h^t x \in Y\}$. Applying (i), we also have $h^t x \in \langle h_Y \rangle x$ whenever $t \leq 0$ and $h^t x \in Y$. Thus $\langle h_Y \rangle x = \langle h \rangle x \cap Y$.

(iv) follows immediately from (iii) and Lemma 2.4. □

Where there is no danger of confusion, we will identify h_Y with the join $h_Y \sqcup \mathrm{id}_{X \smallsetminus Y}$, so that we can regard h_Y as an element of $\tau[h]$.

In this setting, conjugating the induced transformations of h (or more generally, by any element of the centralizer of h) has a predictable effect.

Lemma 2.7. *Let h be a p.a.p. homeomorphism of a compact Hausdorff space X, let Y be a clopen subset of X and let $f \in \mathrm{Homeo}(X)$ such that $[f, h] = \mathrm{id}_X$. Then*

$$f h_Y = h_{fY} f.$$

Proof. Let $x \in X$. If $x \in X \smallsetminus Y$, we see that $f h_Y x = h_{fY} f x = f x$. If $x \in Y$, say that s is the smallest positive integer such that $h^s x \in Y$. Then $f h_Y x = f h^s x$. At the same time, we see that s is the smallest positive integer such that $f h^s x \in fY$; since f and h commute, this is the same as the smallest positive integer s such that $h^s f x \in fY$. Thus $h_{fY} f x = h^s f x = f h^s x$, completing the proof that $f h_Y = h_{fY} f$. □

3. The sign and minimal-periodic partitions

For the next series of lemmas we fix a compact minimal system (X, g) and an element h of $\tau[g]$.

Set $X_p(n)$ to be the set of points x such that $|\langle h \rangle x| = n$, $X_p = \bigcup_{n \geq 1} X_p(n)$ and $X_a := X \smallsetminus X_p$. Lemma 2.1 ensures that X_a is closed, hence compact, and by construction the action of $\langle h \rangle$ on X_a is free (in other words, aperiodic). The idea of the next lemma was suggested to me by F. Le Maître, and goes back to work of R. M. Belinskaya ([1]) in the ergodic theory setting: given $x \in X_a$, then relative to the action of g, there is always a positive or negative 'drift' in the forward iterates of h acting on x.

Lemma 3.1. *Define*

$$X_{h+} := \{x \in X \mid \forall i \in \mathbf{Z}\ \exists n_i \in \mathbf{Z}\ \forall n \geq n_i : c_{g,h^n}(x) > i\};$$
$$X_{h-} := \{x \in X \mid \forall i \in \mathbf{Z}\ \exists n_i \in \mathbf{Z}\ \forall n \geq n_i : c_{g,h^n}(x) < i\}.$$

Then $X_a = X_{h+} \sqcup X_{h-}$, $hX_{h+} = X_{h+}$ and $hX_{h-} = X_{h-}$; moreover, X_{h+} and X_{h-} are both F_σ-sets in X.

Proof. Given $x \in X$, define $\phi_x : \mathbf{Z} \to \mathbf{Z}$ via $\phi_x(n) := c_{g,h^n}(x)$; thus $x \in X_{h+}$ or $x \in X_{h-}$ if and only if $\phi_x(n) \to +\infty$, respectively $\phi_x(n) \to -\infty$, as $n \to +\infty$. Note that the functions ϕ_x satisfy the formula

$$\phi_{hx}(n) = \phi_x(n+1) - \phi_x(1),$$

so the asymptotic behaviour of ϕ_{hx} is the same as that of ϕ_x; it follows that $hX_{h+} = X_{h+}$ and $hX_{h-} = X_{h-}$.

Fix $x \in X$. If x is in a periodic orbit of h, then clearly $x \not\in X_{h+} \cup X_{h-}$; thus we may assume from now on that $x \in X_a$, so that ϕ_x is injective. Let C be the set of $n > 0$ such that one of $\phi_x(n)$ and $\phi_x(n+1)$ is positive and the other is negative. Since h can only act as g^i where $|i| \leq |h|_g$, we must have $0 \leq |\phi_x(n)| \leq |h|_g$ for all $n \in C$. Since ϕ_x is injective, it follows that C is finite. Thus there are only finitely many places at which the value of ϕ_x switches from positive to negative or vice versa, so all but finitely many values of $\phi_x(n)$ for $n \in \mathbf{N}$ have the same sign. Since ϕ_x is injective, if $\phi_x(n)$ is eventually positive, then $\phi_x(n) \to +\infty$ as $n \to +\infty$; otherwise $\phi_x(n)$ is eventually negative, so $\phi_x(n) \to -\infty$ as $n \to +\infty$. This shows that $X_a = X_{h+} \sqcup X_{h-}$. Finally, note that the conditions "$\phi_x(n)$ is eventually greater than 0 as $n \to +\infty$" and "$\phi_x(n)$ is eventually less than 0 as $n \to +\infty$" are F_σ conditions on $x \in X$, so both X_{h+} and X_{h-} are F_σ-sets. □

Lemma 3.2. *Let K be a compact subset of X_a such that $hK = K$. Then K is clopen in X.*

Proof. Let us assume that K is nonempty. We use the sets X_{h+} and X_{h-} to define a descending sequence of closed $\langle h \rangle$-invariant subsets of K recursively, as follows:

Set $K^0 = K$. For each ordinal α where K^α has been defined, write $K^\alpha_+ = K^\alpha \cap X_{h+}$ and $K^\alpha_- = K^\alpha \cap X_{h-}$. Now set $K^{\alpha+1}$ to be the boundary of K^α_+ as a subspace of K^α; equivalently, since K^α is partitioned into K^α_+ and K^α_-, the set $K^{\alpha+1}$ is the boundary of K^α_- as a subspace of K^α. For each limit ordinal $\lambda > 0$ we set $K^\lambda = \bigcap_{\alpha<\lambda} K^\alpha$.

Note that K^α is closed, hence compact, for all ordinals α; the construction also ensures that $hK^\alpha = K^\alpha$. Since we define K^α over all ordinals, the sequence eventually terminates, that is, $K^\alpha = K^{\alpha+1}$; let α be the least ordinal for which this is the case. From the definition of $K^{\alpha+1}$ we see that K^α_+ and K^α_- both have empty interior as subsets of K^α; since K^α_+ and K^α_- are F_σ-sets, they are therefore meagre in K^α. Since $K^\alpha = K^\alpha_+ \cup K^\alpha_-$, it follows by the Baire Category Theorem that K^α is empty. By compactness, we see that α cannot be a limit ordinal, that is, $\alpha = \beta + 1$ for some ordinal β. Thus the boundary of K^β_+ in K^β is empty, in other words, K^β_+ and K^β_- are clopen as subsets of K^β, and in particular they are closed in X. By the minimality of α, the set K^β is nonempty, so at least one of K^β_+ and K^β_- is nonempty.

Suppose that $L := K^\beta_+$ is nonempty. Clearly $hL = L$. For all $x \in L$, there exists $n > 0$ such that $c_{g,h^n}(x) > 0$; furthermore, taking the least such n, we ensure that $0 < c_{g,h^n}(x) \leq |h|_g$. Thus $h^n x = g^t x$ for some $0 < t \leq |h|_g$. We have $h^n L = L$, in particular, $g^t x = h^n x \in L$, so $x \in g^{-t}L$. Letting x vary over L, we see that $L \subseteq \bigcup_{t=1}^{|h|_g} g^{-t}L$, and hence $g^{|h|_g}L \subseteq M$, where $M = \bigcup_{t=0}^{|h|_g-1} g^t L$. We observe that $gM \subseteq M$. Now M is closed and nonempty by construction, so by the minimality of g, we have $M = X$. Thus X is a finite union of $\langle g \rangle$-translates of L. In particular, since L is closed, some $\langle g \rangle$-translate of L has nonempty interior in X, and hence L has nonempty interior in X. A similar argument with g^{-1} in place of g shows that if K^{β}_{-} is nonempty then it has nonempty interior in X. In either case, we conclude that K has nonempty interior in X.

So far we have shown that given a nonempty closed subset K of X_a such that $hK = K$, then K has nonempty interior in X. Now let K' be the boundary of K in X; then K' is a closed subset of X_a with empty interior, such that $hK' = K'$. We conclude that in fact K' must be empty, in other words K is clopen. □

Lemma 3.3. *There is a minimal-periodic partition for* (X, h).

Proof. By Lemma 3.2, X_a is clopen in X, and hence X_p is also clopen. By Lemma 2.1, X_p is partitioned into clopen sets $X_p(n)$.

For each $x \in X_a$, we see that $\overline{\langle h \rangle x}$ is clopen by Lemma 3.2. Let $x, y \in X_a$ and suppose that $Y := \overline{\langle h \rangle x} \cap \overline{\langle h \rangle y} \neq \emptyset$. Then Y is a nonempty open subset of $\overline{\langle h \rangle x}$, so $h^t x \in Y$ for some $t \in \mathbf{Z}$; since Y is closed and $hY = Y$, we then have $\overline{\langle h \rangle x} \subseteq Y$, so in fact $\overline{\langle h \rangle x} = Y$. Similarly, $\overline{\langle h \rangle y} = Y$. Thus by Lemma 2.2, h acts minimally on $\overline{\langle h \rangle x}$ for every $x \in X$. It follows that the closures of $\langle h \rangle$-orbits form a clopen partition of X_a; since X_a is compact, there are only finitely many such orbit closures. This completes the proof that (X, h) admits a minimal-periodic partition. □

Lemma 3.4. *The sets* X_{h+} *and* X_{h-} *are clopen in* X. *Moreover, we have* $X_{h^{-1}+} = X_{h-}$ *and* $X_{h^{-1}-} = X_{h+}$.

Proof. By Lemma 3.2, X_a is clopen in X. Let B be the boundary of X_{h+} in X_a, let $B_+ = B \cap X_{h+}$ and let $B_- = B \cap X_{h-}$. Then B is a compact subset of X_a such that $hB = B$, so B is clopen in X by Lemma 3.2. From the definitions, we now see that B_+ and B_- are F_σ-sets with empty interior in B, hence they are meagre in B. Since $B = B_+ \cup B_-$ it follows that $B = \emptyset$. Thus X_{h+} and X_{h-} are clopen.

Given Lemma 3.1, to show $X_{h^{-1}+} = X_{h-}$ and $X_{h^{-1}-} = X_{h+}$ it is enough to show that $X_{h+} \cap X_{h^{-1}+}$ and $X_{h-} \cap X_{h^{-1}-}$ are empty. Let $Y = X_{h+} \cap X_{h^{-1}+}$; note that Y is clopen by the previous paragraph. Suppose for a contradiction that Y is nonempty. Then we see that $hY = Y$ and that every $\langle h \rangle$-orbit on Y has a unique $\leq_g$-least element. Define a subset Z of Y by setting $x \in Z$ if x is the $\leq_g$-least element of $\langle h \rangle x$, that is, $x \in Y$ and $c_{g,h^n}(x) \geq 0$ for all $n \in \mathbf{Z}$. Then Z is a closed subset of Y and $\langle h \rangle Z = Y$; thus Z has nonempty interior by the Baire Category Theorem. Now let x be an interior point of Z; then $\overline{\langle h \rangle x}$ is topologically perfect (since by Lemma 3.3, it admits a free minimal action of $\langle h \rangle$), so $\langle h \rangle x$ accumulates at x and hence $\langle h \rangle x \cap Z$ is infinite. But the definition of Z ensures that it intersects every $\langle h \rangle$-orbit on Y at exactly one point, so we have a contradiction. From this contradiction we conclude that Y is empty. The proof that $X_{h-} \cap X_{h^{-1}-}$ is empty is similar. □

We now complete the proofs of Theorems 1.2 and 1.4.

Proof of Theorem 1.2. Set h_p to be the restriction of h to X_p; by Lemma 3.3, X_p is clopen and h_p has finite order. By Lemma 3.4, $X_+ := X_{h+}$ and $X_- := X_{h-}$ are both clopen. We thus have a clopen partition of X into $\langle h\rangle$-invariant pieces

$$X = X_p \sqcup X_+ \sqcup X_-,$$

so we can write h as $h = h_p h_+ h_-$ where h_* is the restriction of h to X_* and $h_* \in \tau[h] \le \tau[g]$. Lemma 3.4 ensures that $h_+, (h_-)^{-1} \in \tau_+[g]$. □

Proof of Theorem 1.4. Let g be a homeomorphism of a compact infinite Hausdorff space X such that g admits a minimal-periodic partition; say

$$X = \bigsqcup_{n \in \mathbf{N}} X_p(n) \sqcup \bigsqcup_{i=1}^{m} X_a(i),$$

where each $\langle g\rangle$-orbit on $X_p(n)$ has n points, and each of the spaces $X_a(i)$ is a clopen $\langle g\rangle$-invariant set admitting a free minimal action of $\langle g\rangle$. Let $X_p = \bigsqcup_{n \in \mathbf{N}} X_p(n)$ and let k be the largest natural number such that $X_p(k)$ is nonempty.

Now let $h \in \tau[g]$. Then each of the spaces $X_p, X_a(1), \dots, X_a(m)$ is $\langle h\rangle$-invariant. On X_p, each $\langle g\rangle$-orbit has at most n points, and h preserves the $\langle g\rangle$-orbit relation; it follows that $h^{k!}$ fixes X_p pointwise. Moreover, since $h \in \tau[g]$, we see that for $x \in X_p$, the tuple $(c_{g,h}(x), c_{g,h}(hx), \dots, c_{g,h}(h^{k!-1}x))$ depends continuously on x, and hence $|\langle h\rangle x|$ depends continuously on x; thus we obtain a minimal-periodic partition of (X_p, h). For $1 \le i \le m$ we have a free minimal action of $\langle g\rangle$ on $X_a(i)$, so we can apply Lemma 3.3 to obtain a minimal-periodic partition for $(X_a(i), h)$. Combining the minimal-periodic partitions of (X_p, h) and $(X_a(i), h)$ produces a minimal-periodic partition for (X, h). □

We now give an application of Theorem 1.4 to pointwise almost periodic homeomorphisms.

Corollary 3.5. *Let X be a locally compact Hausdorff space. Suppose that $f \in \mathrm{Homeo}(X)$ is a p.a.p. homeomorphism, and let $f' \in \tau[f]$. Then f' is p.a.p.*

Proof. Let $x \in X$ and let $Y = \overline{\langle f\rangle x}$. Using the fact that f' is a homeomorphism acting locally by powers of f, we observe that $f'Y = Y$ and that the restriction $f' \upharpoonright_Y$ of f' to Y belongs to the topological full group of $(Y, f \upharpoonright_Y)$. The pair $(Y, f \upharpoonright_Y)$ is a compact minimal system by Lemma 2.4 and the fact that f is p.a.p. on X.

We now apply Theorem 1.4, giving a partition of Y into clopen sets $X_a(1), \dots, X_a(m)$ and X_p such that f' acts minimally on $X_a(i)$ and with finite order on X_p. Certainly f' has minimal orbit closures on X_p, and on each of the sets $X_a(i)$ it is minimal. In particular, f' acts minimally on the orbit closure of x. Since $x \in X$ was arbitrary, we conclude by Lemma 2.4 that the action of f' on X is p.a.p. □

4. The relationship between positive and strongly positive elements

Let (X, g) be a compact minimal system. Recall that the set $\tau_>[g]$ of strongly positive elements of $\tau[g]$ consists of those $h \in \tau[g]$ such that $c_{g,h}(x) \ge 0$ for all $x \in X$. Note that this condition immediately implies that $c_{g,h^{-1}}(x) \le 0$ for all $x \in X$.

It is clear that $\tau_>[g] \subseteq \tau_+[g]$; the inclusion is strict in general. The following basic example illustrates the most important distinction between $\tau_>[g]$ and $\tau_+[g]$: in general, $\tau_>[g]$ is closed under multiplication, but not conjugation in $\tau[g]$, whereas $\tau_+[g]$ is closed under conjugation in $\tau_>[g]$, but not under multiplication.

Example 4.1. let $X = \{0,1\}^*$ be the set of infinite binary strings with the pointwise convergence topology and let g act as $g(0w) = 1w$ and $g(1w) = 0g(w)$ for all strings w.

Then (X, g) is a well-known Cantor minimal system, namely the odometer; clearly also $g \in \tau_>[g]$.

(i) Let f be the involution such that $f(0w) = 1w$ and $f(1w) = 0w$ for all strings w, and let $h = fgf$. Then $f \in \tau[g]$; specifically $fx = gx$ for $x \in 0X$ and $fx = g^{-1}x$ for $x \in 1X$. The element h is also minimal, and we have $c_{g,h}(0X) = \{3\}$ and $c_{g,h}(1X) = \{-1\}$, so $h \notin \tau_>[g]$. To see that $h \in \tau_+[g]$, observe that if we follow any forward $\langle h\rangle$-orbit, the values of $c_{g,h}(h^n x)$ are alternately 3 and -1, and hence the overall effect is that $c_{g,h^n}(x) = n - \delta(n)$ where $|\delta(n)| \le 2$.

(ii) Let h' act as g^{-1} on $0X$ and g^3 on $1X$. Then $h' \in \tau_+[g]$ by the same argument as for h, but hh' preserves $0X$ setwise and acts on it as g^{-2}, so $hh' \notin \tau_+[g]$.

We will see over the course of this section that $\tau_+[g]$ consists of exactly the $\tau[g]$-conjugates of $\tau_>[g]$. The situation in Example 4.1(i), where $h \in \tau[g]$ is such that $c_{g,h^n}(x)$ differs from n by a bounded amount over all $(x, n) \in X \times \mathbf{Z}$, turns out to characterize the $\tau[g]$-conjugates of g, and more generally we can understand the cocycle of a positive element as consisting of the cocycle of a strongly positive element plus some bounded perturbation. We will see later (Proposition 5.4 and Theorem 5.6) that $\tau_>[g]$ is the submonoid generated by the induced transformations of g.

Write $\mathrm{N}^+_{\mathrm{Homeo}(X)}(\tau[g])$ for the set of $b \in \mathrm{N}_{\mathrm{Homeo}(X)}(\tau[g])$ such that $b\tau_+[g]b^{-1} = \tau_+[g]$ and $\mathrm{N}^-_{\mathrm{Homeo}(X)}(\tau[g])$ for the set of $b \in \mathrm{N}_{\mathrm{Homeo}(X)}(\tau[g])$ such that $b\tau_+[g]b^{-1} = (\tau_+[g])^{-1}$.

Lemma 4.2. *Let (X, g) be a compact minimal system.*

(i) Given $a \in \tau_+[g]$, then $a^n \in \tau_+[g]$ for all $n \ge 0$.

(ii) We have $\mathrm{N}_{\mathrm{Homeo}(X)}(\tau[g]) = \mathrm{N}^+_{\mathrm{Homeo}(X)}(\tau[g]) \sqcup \mathrm{N}^-_{\mathrm{Homeo}(X)}(\tau[g])$, so $\mathrm{N}^+_{\mathrm{Homeo}(X)}(\tau[g])$ is a subgroup of $\mathrm{N}_{\mathrm{Homeo}(X)}(\tau[g])$ of index at most 2; moreover $\tau[g] \le \mathrm{N}^+_{\mathrm{Homeo}(X)}(\tau[g])$.

Proof. (i) We have $a^0 = \mathrm{id}_X$, which is clearly an element of $\tau_+[g]$. Given $x \in X$ belonging to an infinite orbit of $\langle h\rangle$, then $c_{g,a^i}(x) \to +\infty$ as $i \to +\infty$; given $n > 0$, it then follows immediately that $c_{g,a^{ni}}(x) \to +\infty$ as $i \to +\infty$. Thus $a^n \in \tau_+[g]$ for all $n \ge 0$.

(ii) Let $b \in \mathrm{N}_{\mathrm{Homeo}(X)}(\tau[g])$. Then $\tau[g] = \tau[bgb^{-1}]$. Moreover, since bgb^{-1} acts minimally, we see by Theorem 1.2 that either $bgb^{-1} \in \tau_+[g]$ or $(bgb^{-1})^{-1} \in \tau_+[g]$, but not both.

Suppose that $(bgb^{-1})^{-1} \in \tau_+[g]$ and let $x \in X$. Then $c_{g,(bgb^{-1})^n}(x)$ tends to $-\infty$ as n tends to $+\infty$, and vice versa. It follows that for a fixed $a \in \tau[g]$, then $c_{g,a^n}(x)$ tends to $-\infty$ as n tends to $+\infty$ if and only if $c_{bgb^{-1},a^n}(x)$ tends to $+\infty$ as n tends to $+\infty$, and vice versa. Thus

$$b\tau_+[g]b^{-1} = \tau_+[bgb^{-1}] = (\tau_+[g])^{-1}.$$

Similarly, if $bgb^{-1} \in \tau_+[g]$, we see that $b\tau_+[g]b^{-1} = \tau_+[g]$. Thus $\mathrm{N}_{\mathrm{Homeo}(X)}(\tau[g]) = \mathrm{N}^+_{\mathrm{Homeo}(X)}(\tau[g]) \sqcup \mathrm{N}^-_{\mathrm{Homeo}(X)}(\tau[g])$. In particular, it is clear that either $\mathrm{N}_{\mathrm{Homeo}(X)}(\tau[g]) = \mathrm{N}^+_{\mathrm{Homeo}(X)}(\tau[g])$, or else $\mathrm{N}^+_{\mathrm{Homeo}(X)}(\tau[g])$ is a subgroup of $\mathrm{N}_{\mathrm{Homeo}(X)}(\tau[g])$ of index 2.

Now suppose $b \in \tau[g]$. Then we can bound $c_{g,(bgb^{-1})^n}(x)$ below as follows:

$$c_{g,(bgb^{-1})^n}(x) = c_{g,bg^nb^{-1}}(x) = c_{g,b}(g^n b^{-1}x) + c_{g,g^n}(b^{-1}x) + c_{g,b^{-1}}(x) \ge -2|b|_g + n.$$

In particular, we see that $c_{g,bg^nb^{-1}}(x)$ tends to $+\infty$ as n tends to $+\infty$, so $bgb^{-1} \in \tau_+[g]$, and hence $b\tau_+[g]b^{-1} = \tau_+[g]$. □

Remark 4.3. When X is zero-dimensional, the normalizer of $\tau[g]$ in $\mathrm{Homeo}(X)$ is in fact the whole automorphism group of $\tau[g]$ as a group, because the space can be reconstructed from the group structure. This was stated and proved in the metrizable case in [3], but in fact metrizability is not important for the argument.

Although positive elements of $\tau[g]$ are not strongly positive in general, given $h \in \tau_+[g]$ there is still a large set of points x such that $c_{g,h^n}(x) \geq 0$ for all $n \geq 0$. Given $h \in \tau[g]$, define the **strongly positive domain** of h (with respect to g) to be the set of points $x \in X$ such that $c_{g,h^n}(x) \geq 0$ for all $n \geq 0$.

Lemma 4.4. *Let (X, g) be a compact minimal system, let $h \in \tau_+[g]$, let Y_+ be the strongly positive domain of h with respect to g and let Y_- be the strongly positive domain of h^{-1} with respect to g^{-1}.*

(i) *Y_+ and Y_- are clopen and $X = \bigcup_{i=0}^{k} h^{-i}Y_+ = \bigcup_{i=0}^{k} h^{i}Y_-$ for some finite k.*
(ii) *Let $x \in X$. Suppose $y \in \langle h\rangle x \cap Y_-$ and $z \in \langle h\rangle x \cap Y_+$ are such that $y \leq_g x \leq_g z$. Then $y \leq_h x \leq_h z$.*

Proof. (i) We see that Y_+ is a closed set, since it is defined by a conjunction of conditions on cocycle values. Let X_p be the set of periodic points for h and X_a the set of aperiodic points. Since $h \in \tau_+[g]$, every periodic orbit of h is a singleton; thus $X_p \subseteq Y_+$. On X_a, we note that every forward h-orbit has a $\leq_g$-least point, since $c_{g,h^n}(x) \to +\infty$ as $n \to +\infty$; if y is the $\leq_g$-least point in $\{h^n x \mid n \geq 0\}$, then $y \in Y_+$. Thus Y_+ intersects every forward h-orbit on X, that is, $X = \bigcup_{n=0}^{\infty} h^{-n}Y$. Let $X_a(1), \dots, X_a(m)$ be the infinite minimal orbit closures of h and let Y' be the interior of Y_+ in X. By the Baire Category Theorem, $Y_+ \cap X_a(i)$ has nonempty interior; since $X_a(i)$ is compact and h is minimal on $X_a(i)$, it follows that $X_a(i)$ is covered by finitely many backward h-translates of Y', and hence X_a is covered by finitely many backward h-translates of Y'. Since $X_p \subseteq Y'$, in fact X is covered by finitely many backward h-translates of Y'; say that $X = \bigcup_{i=0}^{k} h^{-i}Y'$ for some nonnegative integer k.

Let $x \in X$ and let $Z_x = \{x, hx, h^2x, \dots, h^k x\}$. Then at least one point $y \in Z_x$ belongs to Y; hence all points in the forward h-orbit of x are $\leq_g$-greater than the $\leq_g$-least element of Z_x, and $x \in Y_+$ if and only if x is the $\leq_g$-least element of Z_x, that is, $c_{g,h^n}(x) \geq 0$ for $0 \leq n \leq k$. This last condition is a clopen condition on x, showing that Y_+ is a clopen subset of X.

The assertions about Y_- follow by the same proof.

(ii) From the definition of Y_-, we see that we cannot have $y >_h x$, as this would imply $y >_g x$. Thus, since $y \in \langle h\rangle x$, we must have $y \leq_h x$. A similar argument shows that $x \leq_h z$. □

Given any $h \in \tau[g]$, there is a unique strongly positive element with the same infinite orbits as h. The most interesting case is when $h \in \tau_+[g]$; in this case, we actually obtain a strongly positive element that is conjugate to h and has the same topological full group.

Definition 4.5. Let (X, g) be a compact minimal system. Given $h \in \tau[g]$, define a function $\pi_> : X \to X$ as follows: if $\langle h\rangle x$ is finite, then $\pi_>(h)(x) = x$, and if $\langle h\rangle x$ is infinite, then $\pi_>(h)(x)$ is the $\leq_g$-least element of $\langle h\rangle x$ such that $x \leq_g \pi_>(h)(x)$.

Lemma 4.6. *Let (X, g) be a compact minimal system and let $h \in \tau[g]$ be such that $[g, h] = \mathrm{id}_X$. Then $h \in \langle g\rangle$.*

Proof. Given $x \in X$, then

$$0 = c_{g,ghg^{-1}h^{-1}}(hgx) = 1 + c_{g,h}(x) + (-1) + c_{g,h^{-1}}(hgx) = c_{g,h}(x) - c_{g,h}(gx);$$

in other words, $c_{g,h}(gx) = c_{g,h}(x)$ for all $x \in X$. Since g acts minimally and $c_{g,h}$ is continuous it follows that $c_{g,h}$ is constant, that is, h is a power of g. □

Proposition 4.7. *Let (X, g) be a compact minimal system; let $h \in \tau_+[g]$, and write $h' = \pi_>(h)$.*

(i) h' is the unique element of $\tau_>[g]$ such that every infinite $\langle h\rangle$-orbit is a $\langle h'\rangle$-orbit and every infinite $\langle h'\rangle$-orbit is a $\langle h\rangle$-orbit.
(ii) Set $\delta(x,t) = c_{g,h^t}(x) - c_{g,(h')^t}(x)$. Then $\delta(x,t)$ is bounded over all $x \in X$ and $t \in \mathbf{Z}$.
(iii) There is an element $k := h^\sigma_{>g}$ of $\tau[h'] \cap \tau_>[g]$, which can be chosen canonically with respect to g and h, such that $h = kh'k^{-1}$; in particular, $\tau[h] = \tau[h']$.
(iv) In the group $\tau[h]$, then h' is the unique conjugate of h that is strongly positive with respect to g. Moreover, $k = h^\sigma_{>g}$ is the unique element of $\tau[h'] \cap \tau_>[g]$ such that $h = kh'k^{-1}$, such that $\mathrm{supp}(k)$ does not contain any nontrivial $\langle h'\rangle$-orbit.

Proof. Let $Y_\pm$ be the strongly positive domain of $h^{\pm1}$ with respect to $g^{\pm1}$.

(i) From the construction of h', it is clear that h' is the unique permutation of X such that every infinite $\langle h\rangle$-orbit is a $\langle h'\rangle$-orbit, every infinite $\langle h'\rangle$-orbit is a $\langle h\rangle$-orbit and $h'x \geq_g x$ for all $x \in X$. In particular, the cocycle $c_{g,h'}$ is well-defined. To prove (i), all that remains is to show that $c_{g,h'}$ is continuous, in other words, locally constant.

By Lemma 4.4(i), Y_+ and Y_- are clopen sets that each intersect every $\langle h\rangle$-orbit. Given $x \in \mathrm{supp}(h)$, let $n_-(x)$ be the $\leq_g$-greatest element of $\langle h\rangle x \cap Y_-$ such that $n_-(x) \leq_g x$, and let $n_+(x)$ be the $\leq_g$-least element of $\langle h\rangle x \cap Y_+$ such that $x <_g n_+(x)$. Then by Lemma 4.4(ii),

$$n_-(x) <_h x \leq_h n_+(x),$$

and hence

$$n_-(x) <_h h'x \leq_h n_+(x),$$

Write $n_\pm(x) = g^{f_\pm(x)}x$. Given Lemma 4.4(i) we see that $f_\pm(x)$ is defined everywhere in X, and also that it is bounded and continuous. Let $I_x = \{i \in \mathbf{Z} \mid x \leq_g h^i x \leq_g n_+(x)\}$, equipped with the ordering that $i \leq j$ if $h^i x \leq_g h^j x$; then I_x is finite for all $x \in X$, and as a function of $x \in X$ it is locally constant. We then see that $c_{g,h'}(x)$ is the successor of 0 in I_x under the given ordering; thus $c_{g,h'}$ is also locally constant as desired, completing the proof of (i).

We also note that, since h and h' have the same orbits and are piecewise powers of the same aperiodic homeomorphism, we have $\tau[h] = \tau[h']$.

(ii) Let us note first that δ has linearly bounded dependence on t: specifically, for any $x \in X$ and $t \in \mathbf{Z}$, we have

$$|\delta(x,t+1) - \delta(x,t)| \leq 2(|h|_g + |h'|_g),$$

where the right-hand side does not depend on (x,t). Note also that if x is a fixed point of h, then $\delta(x,t) = 0$ for all $t \in \mathbf{Z}$, so we do not need to consider such points.

Let $x \in X$ be such that x belongs to an infinite $\langle h\rangle$-orbit, and choose $y \in Y_-$ such that $x \leq_h y$; we then have $x \leq_g y$, and hence $x \leq_{h'} y$. We can choose y as $y = h^k x$, where k is bounded independently of x. In turn, using the fact that $\tau[h] = \tau[h']$, we can write y as $(h')^{k'}x$ where k' is again bounded independently of x.

Now consider $l \geq 0$ such that $h^l y \in Y_+$ and $h^l y \geq_g y$. Let $A = \{z \in \langle h \rangle x \mid y <_g z \leq_g h^l y\}$ and let $B = \{z \in \langle h \rangle x \mid y <_h z \leq_h h^l y\}$. Then $A \subseteq B$ by Lemma 4.4(ii). Clearly $|B| = l$, so $|A| \leq l$. Meanwhile, since h' is strongly positive and $\langle h' \rangle$ has the same orbits as $\langle h \rangle$, we see that $h^l y = (h')^{|A|} y$. Hence

$$(h')^{k'+l} x = (h')^l y \geq_g h^l y = h^{k+l} x,$$

in other words,

$$c_{g,h^{k+l}}(x) - c_{g,(h')^{k'+l}}(x) \leq 0.$$

Since k and k' are bounded independently of x and l, it follows that there is k'' independent of the choices of x and $l \geq 0$ such that

$$\delta(x, l) = c_{g,h^l}(x) - c_{g,(h')^l}(x) \leq k''.$$

The bound above applies for syndetically many values $l \geq 0$. Since $\delta(x, t)$ has linearly bounded dependence on t, it follows that $\delta(x, t)$ is bounded above, uniformly over all of $x \in X$ and $t \geq 0$.

Similarly, we can take $y' \in \langle h \rangle x \cap Y_+$ such that $x \leq_g y'$, and $l \geq 0$ such that $h^l y' \in Y_-$. Let $A' = \{z \in \langle h \rangle x \mid y' <_g z \leq_g h^l y'\}$ and let $B' = \{z \in \langle h \rangle x \mid y' <_h z \leq_h h^l y'\}$. This time $B' \subseteq A'$, and a similar argument to before shows that $\delta(x, t)$ is bounded below, uniformly over all of $x \in X$ and $t \geq 0$. We can extend this to a bound over all $t \in \mathbf{Z}$ by considering the actions of h^{-1} and $(h')^{-1}$ in the same way.

(iii) We can partition X according to the minimal-periodic partition for h, which is the same as the minimal-periodic partition for h', and conjugate independently on each part; thus we may assume h and h' are minimal. We also see in this case that $\tau[h'] \cap \tau_>[g] = \tau_>[h']$. Thus without loss of generality we can take $h' = g$. The expression for $\delta(x, t)$ then simplifies to $\delta(x, t) = c_{g,h^t}(x) - t$, or in other words, $c_{g,h^t}(x) = t + \delta(x, t)$. Using the cocycle formula gives

$$\delta(x, t + s) = \delta(h^s x, t) + \delta(x, s) \tag{1}$$

for all $x \in X$ and $s, t \in \mathbf{Z}$.

Let Z be the set of points $z \in X$ such that $\delta(z, t) \geq 0$ for all $t \in \mathbf{Z}$. It is clear that Z is a subset of Y_+. Moreover, Z intersects every $\langle h \rangle$-orbit: specifically, given $x \in X$ and $s \in \mathbf{Z}$, we see from (1) that

$$h^s x \in Z \Leftrightarrow s \in \operatorname*{arg\,min}_{t \in \mathbf{Z}} \delta(x, t). \tag{2}$$

It is easy to see that δ is a continuous function from $X \times \mathbf{Z}$ to $\mathbf{Z}$, and hence Z is closed. The fact that Z intersects every $\langle h \rangle$-orbit then ensures, via the Baire Category Theorem, that Z has nonempty interior; since h is minimal, we have $X = \bigcup_{i=1}^r h^{-i} Z$ for some natural number r. In other words, given any $x \in X$, the minimum value of $\delta(x, t)$ over all $t \in \mathbf{Z}$ will be achieved for some $1 \leq t \leq r$. By the continuity of δ, we now see that in fact Z is clopen.

The induced transformation h_Z acts on Y_+ with the same orbits as g_Z, but also h_Z is strongly positive with respect to g, and hence with respect to g_Z, since $Z \subseteq Y_+$. The only way this can happen is if $h_Z = g_Z$.

Let $x \in Z$ and let $t = c_{h,h_Z}(x)$. By (2) we have $\delta(x, t) = 0$, in other words, $g^t x = h^t x$; since $g_Z = h_Z$ it follows that $c_{g,g_Z}(x) = t$. Thus g and h have the same return times for Z, that is, $n_g(x, Z) = n_h(x, Z)$ for all $x \in Z$.

We now define a map $k : X \to X$ by setting $kx = h^{s_x} g^{-s_x} x$, where $s_x \in \mathbf{Z}$ is chosen so that $g^{-s_x} x \in Z$. The properties of Z we have established so far ensure that a suitable choice of s_x exists and that all suitable choices will result in the same value of kx. To see that k is bijective, note that given $z \in Z$, then $\langle h \rangle z = \langle g \rangle z$, and given $t \in \mathbf{Z}$ then $kg^t z = h^t z$; it is also easily seen that k acts as a locally constant power of g, so $k \in \tau[g]$. Given $x \in X$, then

$$kgk^{-1}(kx) = kgx = h^{s_x+1}g^{-s_x-1}(gx) = h(h^{s_x}g^{-s_x}x) = hkx;$$

thus $h = kgk^{-1}$. We then see that

$$c_{g,k}(x) = c_{g,h^{s_x}}(g^{-s_x}x) + c_{g,g^{-s_x}}(x) = c_{g,h^{s_x}}(g^{-s_x}x) - s_x = \delta(g^{-s_x}x, s_x) \geq 0.$$

Thus $k \in \tau_>[g]$.

(iv) By part (i), h' is strongly positive with respect to g, and by part (iii), h' is conjugate to h in $\tau[h]$. Conversely, given a conjugate h'' of h in $\tau[h]$ that is strongly positive, then for every $x \in X$, we have $\langle h' \rangle x = \langle h \rangle x = \langle h'' \rangle x$ and both $h'x$ and $h''x$ are the $\leq_g$-least element y of $\langle h \rangle x$ such that $x <_g y$; hence $h' = h''$.

Let $k = h^{\sigma}_{>g}$ be constructed as in part (iii), and write Z' for the set of fixed points of k; since $k \in \tau[g]$, Z' is clopen. From the construction, we see that Z' intersects every $\langle h \rangle$-orbit; thus $\mathrm{supp}(k)$ does not contain any nontrivial $\langle h \rangle$-orbit, or equivalently, $\mathrm{supp}(k)$ does not contain any nontrivial $\langle h' \rangle$-orbit. Suppose that k' is some other element of $\tau[h'] \cap \tau_>[g]$, such that $h = k'h'(k')^{-1}$ and such that $\mathrm{supp}(k')$ does not contain any nontrivial $\langle h' \rangle$-orbit. Then $k' = kl$ where $l \in \tau[h']$ and l centralizes h'. It follows by Lemma 4.6 that l acts on each of the minimal parts of h' as a power of h'. Given $x \in Z'$, then

$$0 \leq c_{g,k'}(l^{-1}x) = c_{g,k}(x) + c_{g,l}(l^{-1}x) = c_{g,l}(l^{-1}x);$$

since h' is strongly positive, this implies $c_{h',l}(l^{-1}x) \geq 0$, so on the minimal part of h' containing $l^{-1}x$, then l acts as a nonnegative power of h'. Since $l^{-1}Z'$ intersects every minimal part of h', it follows that in fact $c_{h',l}(x) \geq 0$ for all $x \in X$, so l is strongly positive. On the other hand, given $x \in \mathrm{supp}(h')$ such that x is fixed by k', then

$$0 = c_{g,k'}(x) = c_{g,k}(lx) + c_{g,l}(x);$$

since k and l are strongly positive, this is only possible if $c_{g,k}(lx) = c_{g,l}(x) = 0$. Thus l has a fixed point on every nontrivial orbit of $\langle h' \rangle$; since $\langle h' \rangle$ acts transitively on each $\langle h \rangle$-orbit, and l acts on each $\langle h \rangle$-orbit as a power of h', it follows that $l = \mathrm{id}_X$, and hence $k = k'$. □

Here is the characterization of the map $\pi_>$ for arbitrary elements of $\tau[g]$.

Corollary 4.8. *Let (X, g) be a compact minimal system; let $h \in \tau_+[g]$, and write $h' = \pi_>(h)$. Then h' is the unique element of $\tau_>[g]$ such that every infinite $\langle h \rangle$-orbit is a $\langle h' \rangle$-orbit and every infinite $\langle h' \rangle$-orbit is a $\langle h \rangle$-orbit.*

Proof. Apply the sign partition to h: we have a partition of X into clopen parts X_p, X_+ and X_- such that, writing h_* for the restriction of h to X_*, then h_p has finite order and $h_+, (h_-)^{-1} \in \tau_+[g]$. It is then clear that we can partition $\pi_>(h)$ in a similar manner: given $x \in X$, then

$$\pi_>(h)(x) = \begin{cases} x & \text{if } x \in X_p \\ \pi_>(h_+)(x) & \text{if } x \in X_+ \\ \pi_>((h_-)^{-1})(x) & \text{if } x \in X_- \end{cases}.$$

The conclusion is now clear from Proposition 4.7. □

Given a compact minimal system (X, g), we can now define the **strong sign form** of $h \in \tau[g]$. Start with the sign partition $h = h_p h_+ h_-$. We now make the substitutions

$$h_+ = h^\sigma_> h_> (h^\sigma_>)^{-1} \text{ and } h_- = h^\sigma_< h_< (h^\sigma_<)^{-1},$$

where the factors are as follows: $h_> = \pi_>(h_+)$; $h_< = (\pi_>((h_-)^{-1}))^{-1}$; $h^\sigma_> = (h_+)^\sigma_{>g}$; and $h^\sigma_< = (((h_-)^{-1})^\sigma_{>g})^{-1}$. Proposition 4.7 ensures that these factors are well-defined and the substitution results in the same homeomorphism of X, and moreover that $h_>, h^\sigma_> \in \tau_>[g]$ and $h_<, h^\sigma_< \in (\tau_>[g])^{-1}$.

The following is an immediate consequence of Lemma 4.2 and Proposition 4.7.

Corollary 4.9. *Let (X, g) be a compact minimal system. Then*

$$\tau_+[g] = \bigcup_{k \in \tau[g]} k\tau_>[g]k^{-1}.$$

An interesting special case of Proposition 4.7 is when $\pi_>(h) = g$. This situation has several equivalent characterizations.

Proposition 4.10. *Let (X, g) be a compact minimal system and let $h \in \tau[g]$. Then the following are equivalent:*

(i) $\pi_>(h) = g$;
(ii) h *is* $\tau[g]$*-conjugate to* g *or* g^{-1};
(iii) $\tau[h] = \tau[g]$;
(iv) $\langle h \rangle$ *acts transitively on every* $\langle g \rangle$*-orbit;*
(v) there exists $\epsilon \in \{0, 1\}$ *such that* $c_{g,h^t}(x) - (-1)^\epsilon t$ *is bounded over all* $x \in X$ *and* $t \in \mathbf{Z}$.

Proof. Suppose (i) holds. By Corollary 4.8 we see that h must act minimally on X; in light of the sign partition, this means exactly one of h and h^{-1} belongs to $\tau_+[g]$. We then deduce (ii) from Proposition 4.7(iii) and (v) from Proposition 4.7(ii).

The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are clear.

If (iv) holds, then the only strongly positive element of $\tau[g]$ with the same infinite two-sided orbits as h is g itself; thus (i) holds. This completes the proof that (i)-(iv) are equivalent.

Suppose (v) holds. We immediately see that every $\langle h \rangle$-orbit is infinite, and that for all sufficiently large $s \geq 0$, we have

$$\forall x \in X, t \in \left[-\frac{2s}{3}, \frac{2s}{3}\right] : \; h^t x \in \{g^{-s}x, \dots, g^{-1}x, x, gx, \dots, g^s x\};$$

it follows that for every $x \in X$, at least two thirds of the points in the interval $\{g^{-s}x, \dots, g^s x\}$ belong to $\langle h \rangle x$. This ensures that $\langle h \rangle$ acts transitively on every $\langle g \rangle$-orbit. Thus (v) implies (iv); we have already seen that (i)-(iv) are equivalent and that (i) implies (v), so the proof that all five statements are equivalent is complete. □

In particular, if (X, g) is a compact minimal system, then none of the $\tau[g]$-conjugates of g other than g itself are strongly positive with respect to g, whereas all of them are positive in the weaker sense. Thus we can only have $\tau_+[g] = \tau_>[g]$ in the degenerate case that g is central in $\tau[g]$; considering induced transformations, this can only occur if X is connected, in which case $\tau[g] = \langle g \rangle$.

We also obtain the following decomposition of the normalizer of $\tau[g]$ in Homeo(X).

Proposition 4.11. *Let (X, g) be a compact minimal system and let $H = \mathrm{Homeo}(X)$. Then*

$$\mathrm{N}_H(\tau[g]) = \tau[g]\mathrm{C}_H(g)\langle h\rangle,$$

where either $h = \mathrm{id}_X$ (in the case that g and g^{-1} are not conjugate in H), or h is an element such that $hgh^{-1} = g^{-1}$ and $h^2 \in \tau[g]\mathrm{C}_H(\tau[g])$ (in the case that g and g^{-1} are conjugate in H). Moreover, $\tau[g] \cap \mathrm{C}_H(g) = \langle g\rangle$.

Proof. Let $K = \mathrm{N}^+_H(\tau[g])$; by Lemma 4.2, K is a subgroup of $\mathrm{N}_H(\tau[g])$ of index at most 2 and $\tau[g] \le K$. From the definition we see that g is not conjugate to g^{-1} in K. Let C be the conjugacy class of g in K. Then $\tau[g'] = \tau[g]$ for all $g' \in C$. By Proposition 4.10, it follows that C is in fact the conjugacy class of g in $\tau[g]$; hence $K = \tau[g]\mathrm{C}_K(g)$. We observe moreover that the centralizer of g in H also normalizes the topological full group of g and preserves the orientation, so in fact $K = \tau[g]\mathrm{C}_H(g)$.

If g and g^{-1} are not conjugate in H, then $\mathrm{N}_H(\tau[g]) = K$ and we have the required decomposition. Otherwise, there is $h \in H$ such that $hgh^{-1} = g^{-1}$. We see that h normalizes $\tau[g]$ but does not lie in K, so $\mathrm{N}_H(\tau[g]) = K\langle h\rangle$ and $h^2 \in K$. □

In contrast to $\tau_+[g]$, the set $\tau_>[g]$ is a submonoid of $\tau[g]$.

Lemma 4.12. *Let X be an infinite compact Hausdorff space and let $g \in \mathrm{Homeo}(X)$ be aperiodic. Then $\tau_>[g]$ is a submonoid of $\tau[g]$. Moreover, given $a, b \in \tau[g]$, then $ba^{-1} \in \tau_>[g]$ if and only $c_{g,a}(x) \le c_{g,b}(x)$ for all $x \in X$.*

Proof. Certainly $\tau_>[g]$ contains the identity homeomorphism $1 \in \tau[g]$. Given $a, b \in \tau[g]$ and $x \in X$, then

$$c_{g,ab}(x) = c_{g,a}(bx) + c_{g,b}(x);$$

thus if $c_{g,a}$ and $c_{g,b}$ only take nonnegative values, then so does $c_{g,ab}$. Hence $\tau_>[g]$ is a submonoid of $\tau[g]$.

Given $a, b \in \tau[g]$, we see that

$$c_{g,ba^{-1}}(x) = c_{g,b}(a^{-1}x) + c_{g,a^{-1}}(x) = c_{g,b}(a^{-1}x) - c_{g,a}(a^{-1}x).$$

Thus $c_{g,ba^{-1}}(x) \ge 0$ for all $x \in X$ (in other words, $ba^{-1} \in \tau_+[g]$) if and only if $c_{g,a}(a^{-1}x) \le c_{g,b}(a^{-1}x)$ for all $x \in X$, or in other words, $c_{g,a}(x) \le c_{g,b}(x)$ for all $x \in X$. □

One easy consequence of Lemma 4.12 is that the right $\langle g\rangle$-translates of $\tau_>[g]$ cover $\tau[g]$.

Corollary 4.13. *Let (X, g) be a compact minimal system and let $a_1, \dots, a_n \in \tau[g]$. Then there is $k \in \mathbf{N}$ such that $a_i g^k \in \tau_>[g]$ for all $i \in \{1, \dots, n\}$.*

Proof. By Lemma 4.12, it suffices to take $k \in \mathbf{N}$ such that $k \ge \max\{|a_1|_g, \dots, |a_n|_g\}$. □

5. A permutational construction of the index map

Definition 5.1. Let (X, g) be a compact minimal system. An **index map** for (X, g) is a group homomorphism $I_g : \tau[g] \to \mathbf{Z}$ such that $I_g(g) = 1$ and $I_g(h) > 0$ for all $h \in \tau_>[g] \smallsetminus \{\mathrm{id}_X\}$.

When the homeomorphism g is clear from context, we will write I in place of I_g. In this section, we will show that every compact minimal system admits a unique index map in the sense defined. In itself this is not new: a similar result for Cantor minimal systems was given in [3], where the index map was defined by an integral formula, and the index map also has a homological generalization to étale groupoids, due to Matui ([7, §7]). The main purpose of this section is rather to present a new construction of the index map for a compact minimal system that is natural from a permutational perspective and yields some additional information about the structure of elements of $\tau[g]$.

For the uniqueness of the index map, we show that if (X, g) admits an index map I, then $I(g_A) = 1$ for every nonempty clopen subset A of X and that $\tau[g]$ is generated as a group by induced transformations. Here we mimic the approach of Le Maître in [6, §4.3].

Lemma 5.2 (See [6, Lemma 4.16]). *Let (X, g) be a compact minimal system and let $h \in \tau_>[g]$. Then $hg^{-1}_{\mathrm{supp}(h)} \in \tau_>[g]$.*

Proof. Note that by Theorem 1.2, the set $\mathrm{supp}(h)$ is clopen; since h is strongly positive we have $c_{g,h}(x) > 0$ for all $x \in \mathrm{supp}(h)$. Let $x \in X$ and write $g_{\mathrm{supp}(h)}x = g^s x$ and $hx = g^t x$. If $x \notin \mathrm{supp}(h)$ then $s = t = 0$. If $x \in \mathrm{supp}(h)$, then by definition s is the least positive integer such that $g^s x \in \mathrm{supp}(h)$, whereas t is some positive integer such that $g^t x \in \mathrm{supp}(h)$. In either case, $s \leq t$, and hence $hg^{-1}_{\mathrm{supp}(h)} \in \tau_>[g]$ by Lemma 4.12. □

Lemma 5.3. *Let (X, g) be a compact minimal system and let $\alpha \in \mathrm{Hom}(\tau[g], \mathbf{Z})$. Then $\alpha(g_A) = \alpha(g)$ for every nonempty clopen subset A of X.*

Proof. Given a nonempty clopen set A, we see that $(g_A)^{-1}g$ has finite order: Specifically, there exists $k \in \mathbf{N}$ such for every $x \in X$, at least one of the points $gx, g^2x, \dots, g^kx$ belongs to A, and then it follows that every orbit of $(g_A)^{-1}g$ has length at most k. In particular, since $\mathbf{Z}$ is torsion-free we have $\alpha((g_A)^{-1}g) = 0$ and hence

$$\alpha(g_A) = \alpha(g).$$ □

Proposition 5.4 (See also [6, Proposition 4.17]). *Let (X, g) be a compact minimal system admitting an index map I.*

(i) *Let $h \in \tau[g]$. Let r be the minimum value of $c_{g,h}$. Set $A_1 = \mathrm{supp}(hg^{-r})$, and thereafter*
$$A_{i+1} = \mathrm{supp}(hg^{-r}g^{-1}_{A_1}g^{-1}_{A_2}\dots g^{-1}_{A_i}).$$
Then $I(h) \geq r$; A_1 is a proper clopen subset of X; $A_{i+1} \subseteq A_i$ for all $i \geq 1$; and $A_i = \emptyset$ if and only if $i > I(h) - r$. In particular,
$$h = g_{A_{I(h)-r}} \dots g_{A_2}g_{A_1}g^r,$$
where we interpret the empty word as id_X.

(ii) *We have $\mathrm{Hom}(\tau[g], \mathbf{Z}) \cong \mathbf{Z}$ and the index map is unique.*

Proof. (i) We see that the minimum value of $c_{g,hg^{-r}}$ is 0; thus $hg^{-r} \in \tau_>[g]$ and the support of hg^{-r} is properly contained in X. We may assume that $h \neq g^r$, otherwise the conclusion is trivial. The hypotheses on the index map then ensure that

$$0 < I(hg^{-r}) = I(h) - r.$$

By Lemma 5.3 we have $I(g_A) = 1$ for every nonempty clopen subset A of X. Thus hg^{-r} cannot be expressed as a product of fewer than $I(h) - r$ induced transformations of g, and hence A_i is nonempty for all $i \leq I(h) - r$. We see that A_{i+1} is the support of the product of two elements that are both supported on A_i; thus $A_{i+1} \subseteq A_i$. Now let

$$h' = g_{A_{I(h)-r}} \dots g_{A_2}g_{A_1}g^r; \quad d = h(h')^{-1}.$$

Using the fact that I is a homomorphism, we see that $I(d) = 0$; by repeated application of Lemma 5.2, we have $d \in \tau_>[g]$. Thus $d = \mathrm{id}_X$, so $h' = h$. In particular, A_i is empty for all $i > I(h) - r$.

(ii) By part (i), $\tau[g]$ is generated as a group by induced transformations of g. The fact that $\mathrm{Hom}(\tau[g], \mathbf{Z}) \cong \mathbf{Z}$ now follows from Lemma 5.3. In particular, I is determined as a homomorphism by the fact that $I(g) = 1$, so the index map is unique. □

For the existence of the index map, we use the orbits of elements of $\tau[g]$. In particular, given $h \in \tau[g]$, then every $\langle g \rangle$-orbit on X contains a fixed number of infinite $\langle h \rangle$-orbits, which we can further distinguish by their orientation.

Proposition 5.5. *Let (X, g) be a compact minimal system and let $h \in \tau[g]$. Then there are natural numbers $n_+ := o^+_g(h)$ and $n_- := o^-_g(h)$ such that for all $x \in X$, then h has exactly n_+ nontrivial positive orbits and n_- nontrivial negative orbits on $\langle g \rangle x$. Moreover, the following holds:*

(i) We have $o^\pm_g(h) = o^\pm_g(khk^{-1})$ for all $k \in \mathrm{N}^+_{\mathrm{Homeo}(X)}(\tau[g])$.

(ii) We have

$$o^\pm_g(h) \geq o^\pm_g(h_A) = o^\pm_{g_A}(h_A)$$

for all clopen subsets A of X, with $o^\pm_g(h) = o^\pm_g(h_A)$ if and only if A intersects every infinite $\langle h \rangle$-orbit of the corresponding orientation.

Proof. Write $o^x_g(h)$ for the number of infinite $\langle h \rangle$-orbits on $\langle g \rangle x$ (we allow $o^x(h) = +\infty$ for the moment). Given a nonempty clopen subset A of X, then $\langle g \rangle x \cap A$ is a $\langle g_A \rangle$-orbit, and $\langle h_A \rangle$ acts transitively on $\langle h \rangle y \cap A$ for every $y \in X$ such that $\langle h \rangle y \cap A$ is nonempty. Given $y \in X$ such that $\langle h \rangle y$ is infinite, then h acts minimally on $\overline{\langle h \rangle y}$, by Theorem 1.4; in particular, $\overline{\langle h \rangle y}$ is a perfect space, so $\langle h \rangle y \cap A$ is either empty or infinite. Thus if $x \in Y$, then

$$o^x_g(h) \geq o^x_{g_A}(h_A) = o^x_{g_A}(h_A);$$

if $o^x_g(h_A)$ is finite, then $o^x_g(h) = o^x_g(h_A)$ if and only if A intersects every infinite $\langle h \rangle$-orbit on $\langle g \rangle x$.

Suppose for the moment that $h \in \tau_>[g]$; let Y be the support of h.

Observe that every $\langle g_Y \rangle$-orbit is a disjoint union of $\langle h \rangle$-orbits and that every $\langle h \rangle$-orbit on Y is infinite. For any $x \in Y$, every $\langle h \rangle$-orbit on the set $\langle g_Y \rangle x$ passes through the finite set $\{g_Y x, g^2_Y x, \dots, g^{|h|_g}_Y x\}$. Thus $o^x_g(h) = o^x_{g_Y}(h)$ is finite for all $x \in Y$.

Given $x \in Y$, we can compute $o^x_g(h)$ by counting the number of connected components in the finite graph Γ_x, where $V\Gamma_x$ is the interval $\{g_Y, g^2_Y, \dots, g^{|h|_g}_Y\}$, and we draw an edge from g^i to g^j if $h(g^i x) = g^j x$. We see that the vertices are independent of x, and the edges each depend continuously on x; thus $o^x_g(h)$ depends continuously on $x \in Y$. A similar argument shows that $o^x_g(h)$ depends continuously on x for $x \in g^t Y$, for any $t \in \mathbf{Z}$; thus $o^x_g(h)$ depends continuously on $x \in X$. Since $\langle g \rangle gx = \langle g \rangle x$, we have $o^x_g(h) = o^{gx}_g(h)$. Since g is minimal, we deduce that $o^x_g(h) = o^y_g(h)$ for all $x, y \in X$; since $h \in \tau_>[g]$, every $\langle h \rangle$-orbit is positive. Thus so we can define $o^+_g(h) := o^x_g(h)$ for any $x \in X$ and $o^-_g(h) = 0$.

Given $x \in X$ and $k \in \mathrm{N}^+_{\mathrm{Homeo}(X)}(\tau[g])$, then $\tau_+[g] = \tau_+[kgk^{-1}]$, so $\langle kgk^{-1} \rangle$ has the same orbits as $\langle g \rangle$ with the same orientation; thus

$$o^+_g(khk^{-1}) = o^+_{kgk^{-1}}(khk^{-1}) = o^+_g(h).$$

This completes the proof of the proposition in the case that $h \in \tau_>[g]$. By Corollary 4.9, we immediately deduce the proposition for $h \in \tau_+[g]$.

For the general case, we take the sign partition $h = h_p h_+ h_-$ of h, with corresponding partition $X = X_p \sqcup X_+ \sqcup X_-$ of X. We now observe that the number of infinite positive $\langle h \rangle$-orbits on $\langle g \rangle x$ for $x \in X$ is exactly $o_g(h_+)$, and the number of infinite negative $\langle h \rangle$-orbits is exactly $o_{g^{-1}}(h_-)$. Now define $o^+_g(h) := o_g(h_+)$ and $o^-_g(h) := o^+_{g^{-1}}(h_-)$. Since $h_+ \in \tau_+[g]$ and $h_- \in \tau_+[g^{-1}]$, the desired conclusions all follow from the positive case of the proposition. $\square$

With respect to a given compact minimal system (X, g), and given $h \in \tau[g]$, we define the **positive orbit number** of h to be $o^+_g(h)$, the **negative orbit number** of h to be $o^-_g(h)$, and the **orbit number** to be $o_g(h) := o^+_g(h) + o^-_g(h)$. We will omit the subscript g when the defining minimal homeomorphism is clear from context.

We can now state the main theorem of this section; the proof will proceed via a series of lemmas. The formula given for the index map as an average of cocycle values is an analogue of the integral formula in [3].

Theorem 5.6. *Let (X, g) be a compact minimal system. Then (X, g) admits a unique index map $I : \tau[g] \to \mathbf{Z}$, satisfying the equations*

$$I(h) = o^+(h) - o^-(h) = \lim_{l \to \infty} \frac{1}{l} \sum_{1 \le j \le l} c_{g,h}(g^j x) \ (\forall x \in X).$$

From now until the proof of Theorem 5.6, we define

$$I^\pi(h) = o^+(h) - o^-(h).$$

Our first observation is that I^π satisfies the positivity condition required for the index map.

Lemma 5.7. *Let (X, g) be a compact minimal system and let $h \in \tau_+[g]$. Then $I^\pi(h) \ge 0$, with $I^\pi(h) = 0$ if and only if $h = \mathrm{id}_X$.*

Proof. If $h = \mathrm{id}_X$ then certainly $I^\pi(h) = 0$, so we may assume h is nontrivial. Since $h \in \tau_+[g]$ we have $h = h_+$ and $h_- = \mathrm{id}_X$; thus $I^\pi(h) = o(h)$. Moreover, h has at least one infinite orbit, ensuring that $o(h) \ge 1$. $\square$

The next two lemmas prove the formula for I^π as an average of cocycle values.

Lemma 5.8. *Let X be a compact Hausdorff space and let g be a minimal homeomorphism of X. Then for all $x \in X$ and $h \in \tau_>[g]$, the following formula holds:*

$$o(h) = \lim_{l \to \infty} \frac{1}{l} \sum_{1 \le j \le l} c_{g,h}(g^j x).$$

Proof. Fix $h \in \tau_>[g]$, $k \ge 2|h|_g$ and $x \in X$. Define a graph Γ as follows: $V\Gamma = \{g^i x \mid 0 \le i \le k\}$, and we place an edge from $g^i x$ to $g^j x$ if $g^j x = h(g^i x)$. The choice of k ensures that every nontrivial $\langle h \rangle$-orbit on $\langle g \rangle x$ is represented by at least two vertices of Γ, and by Proposition 5.5, the number $o(h)$ of nontrivial $\langle h \rangle$-orbits on $\langle g \rangle x$ does not depend on x. Since h acts only by nonnegative powers of g, the number of nontrivial $\langle h \rangle$-orbits on $\langle g \rangle x$ is the number of nontrivial connected components of Γ (where 'trivial connected component' means an isolated vertex). Each nontrivial connected component has exactly one terminal vertex, that is, a vertex y such that $hy \notin V\Gamma$; given such a y, then $hy = g^t x$ for $t > k$. We can thus compute $o(h)$ as

$$o(h) = \sum_{0 \leq i \leq k} \chi(g^i x, k - i),$$

where $\chi(x,t) = 1$ if $c_{g,h}(x) > t$ and $\chi(x,t) = 0$ otherwise.

Since the left-hand side does not depend on x, we are free to average over the forward g-orbit:

$$o(h) = \frac{1}{l} \sum_{1 \leq j \leq l} \sum_{0 \leq i \leq k} \chi(g^i(g^j x), k - i) = \sum_{0 \leq i \leq k} \left(\frac{1}{l} \sum_{1 \leq j \leq l} \chi(g^{i+j} x, k - i) \right).$$

Now fix i and let l tend to infinity. Then the difference

$$\left(\frac{1}{l} \sum_{1 \leq j \leq l} \chi(g^{i+j} x, k - i) \right) - \left(\frac{1}{l} \sum_{1 \leq j \leq l} \chi(g^j x, k - i) \right)$$

tends to zero, because the two sums differ only by a bounded number of summands. Thus

$$o(h) = \sum_{0 \leq i \leq k} \lim_{l \to \infty} \left(\frac{1}{l} \sum_{1 \leq j \leq l} \chi(g^j x, k - i) \right).$$

On the right-hand side, since the outer sum has finitely many terms, we can reverse the order of summation again:

$$o(h) = \lim_{l \to \infty} \left(\frac{1}{l} \sum_{1 \leq j \leq l} \left(\sum_{0 \leq i \leq k} \chi(g^j x, k - i) \right) \right)$$

Since $c_{g,h}$ only takes integer values in the interval $(0, k]$, we can simplify the innermost sum on the right-hand side:

$$o(h) = \lim_{l \to \infty} \frac{1}{l} \sum_{1 \leq j \leq l} c_{g,h}(g^j x). \qquad \square$$

Lemma 5.9. *Let X be a compact Hausdorff space and let g be a minimal homeomorphism of X. Then for all $x \in X$ and $h \in \tau[g]$, the following formula holds:*

$$I^\pi(h) = \lim_{l \to \infty} \frac{1}{l} \sum_{1 \leq j \leq l} c_{g,h}(g^j x).$$

Proof. We use the sign partition of h; let $X = X_p \sqcup X_+ \sqcup X_-$ be the corresponding partition of X, let $x \in X$ and let χ_* be the indicator function of X_*. If $O \subseteq X_p$ is a union of $\langle h \rangle$-orbits, we see that

$$\sum_{y \in O} c_{g,h}(y) = 0;$$

since h_p has finite order, it then follows easily that

$$0 = \lim_{l \to \infty} \frac{1}{l} \sum_{1 \leq j \leq l} c_{g,h}(g^j x) \chi_p(g^j x). \tag{3}$$

Now let A_+ be the strongly positive domain of h_+. Then by Proposition 5.5 and Lemma 4.4, we have $h_{A_+} = (h_+)_{A_+}$ and $o_g(h_+) = o_g(h_{A_+})$. By Lemma 5.8, it follows that

$$o_g(h_+) = \lim_{l\to\infty} \frac{1}{l} \sum_{1\le j\le l} c_{g,h_{A_+}}(g^j x).$$

Define an h_+-circuit be a set of the form $R = \{y, hy, h^2y, \dots, h^{k-1}y\}$ for some $k \ge 1$ such that $y, h^k y \in A_+$ but $h^{k'}y \notin A_+$ for $1 \le k' < k$. Then $h^k y = h_{A_+}y$, while the points $hy, h^2y, \dots, h^{k-1}y$ are fixed by h_{A_+}; it follows that

$$\sum_{z\in R} c_{g,h}(z) = c_{g,h^k}(y) = c_{g,h_{A_+}}(y) = \sum_{z\in R} c_{g,h_{A_+}}(z).$$

Now the set $X_+ \cap \langle g\rangle x$ is a disjoint union of h_+-circuits. Moreover, we see that since h is aperiodic and p.a.p. on X_+, the cardinality of h_+-circuits is bounded, and hence there is some r such that for every $y \in X_+ \cap \langle g\rangle x$, we have $g^{-r}y \le_g z \le_g g^r y$ for all z in the h_+-circuit containing y. Thus the intersection $X_+ \cap \{gx, g^2x, \dots, g^l x\}$ consists of a union of h_+-circuits plus a bounded number of additional points, ensuring that the difference

$$\sum_{1\le j\le l} c_{g,h_{A_+}}(g^j x) - \sum_{1\le j\le l} c_{g,h}(g^j x)\chi_+(g^j x)$$

is bounded independently of l. We conclude that

$$o^+(h) = o_g(h_+) = \lim_{l\to\infty} \frac{1}{l} \sum_{1\le j\le l} c_{g,h}(g^j x)\chi_+(g^j x). \tag{4}$$

By a similar argument,

$$o_{g^{-1}}(h_-) = \lim_{l\to\infty} \frac{1}{l} \sum_{1\le j\le l} c_{g^{-1},h}(g^j x)\chi_-(g^j x) = -\lim_{l\to\infty} \frac{1}{l} \sum_{1\le j\le l} c_{g,h}(g^j x)\chi_-(g^j x);$$

noting that $o_{g^{-1}}(h_-) = o^-(h)$, we end up with

$$-o^-(h) = \lim_{l\to\infty} \frac{1}{l} \sum_{1\le j\le l} c_{g,h}(g^j x)\chi_-(g^j x). \tag{5}$$

Adding (3), (4) and (5) together, we obtain

$$I^\pi(h) := o^+(h) - o^-(h) = \lim_{l\to\infty} \frac{1}{l} \sum_{1\le j\le l} c_{g,h}(g^j x).$$ □

We can now prove the theorem.

Proof of Theorem 5.6. The formula for I^π as an average of cocycle values was proved in Lemma 5.9. By Proposition 5.4 there is at most one index map for (X, g), so it suffices to show that I^π is an index map. It is clear that $I^\pi(g) = 1$, and by Lemma 5.3 we have $I^\pi(h) > 0$ for all $h \in \tau_>[g] \setminus \{\mathrm{id}_X\}$. All that remains is to show that I^π is a homomorphism.

Let $a, b \in \tau[g]$. It is clear from the definition that $I^\pi(a^{-1}) = -I^\pi(a)$. By Lemma 5.9, for all $x \in X$ we have

$$I^\pi(ab) = \lim_{l\to\infty} \frac{1}{l} \sum_{1\leq j\leq l} c_{g,ab}(g^j x) = \lim_{l\to\infty} \frac{1}{l}\left(\sum_{1\leq j\leq l} c_{g,a}(bg^j x) + \sum_{1\leq j\leq l} c_{g,b}(g^j x)\right)$$
$$= \lim_{l\to\infty} \frac{1}{l} \sum_{1\leq j\leq l} c_{g,a}(bg^j x) + I^\pi(b).$$

Up to reordering, the summands of $\sum_{1\leq j\leq l} c_{g,a}(bg^j x)$ and $\sum_{1\leq j\leq l} c_{g,a}(g^j x)$ are the same, with at most $4|b|_g$ exceptions. Since $c_{g,a}$ only takes finitely many values, the summands are also bounded, and thus

$$\lim_{l\to\infty} \frac{1}{l} \sum_{1\leq j\leq l} c_{g,a}(bg^j x) = \lim_{l\to\infty} \frac{1}{l} \sum_{1\leq j\leq l} c_{g,a}(g^j x) = I^\pi(a).$$

Hence $I^\pi(ab) = I^\pi(a) + I^\pi(b)$, completing the proof. □

Remark 5.10. As an alternative proof, one can take the formula of Lemma 5.9, choose a Borel probability measure μ with respect to which g is measure-preserving and ergodic, and apply the pointwise ergodic theorem; the conclusion is that I^π is given by exactly the same integral formula as in [3], namely

$$I^\pi(h) = \int_X c_{g,h}(x)\, \mathrm{d}\mu(x).$$

We conclude this section with some further properties of the orbit number.

The fact that the index map is a homomorphism puts some restrictions on the additivity of the orbit number for positive elements.

Corollary 5.11. *Let (X, g) be a compact minimal system, let $h, h' \in \tau_+[g]$, and define the orbit number with respect to g. Then*

$$o(hh') \geq o(h) + o(h'),$$

with equality if and only if $(hh')^t \in \tau_+[g]$ for some $t > 0$.

Proof. Since h and h' are positive, we see that $I(h) = o(h) = o^+(h)$ and $I(h') = o(h') = o^+(h')$. By Theorem 5.6 we have $o^+(hh') = I(hh') + o^-(hh')$, and clearly $o(hh') \geq o^+(hh')$. Thus

$$o(hh') \geq o^+(hh') \geq I(hh') = o(h) + o(h').$$

We see that equality holds if and only if $o^-(hh') = 0$. Given the sign partition of hh', we have $o^-(hh') = 0$ if and only if the negative part of hh' is empty; equivalently, $(hh')^t$ is positive, where t is the order of the periodic part of hh'. □

As an illustration, let g, h and h' be as in Example 4.1(ii). Then h and h' are positive with $o_g(h) = o_g(h') = 1$, but hh' has a nonempty negative part, with $o_g^+(hh') = 3$ and $o_g^-(hh') = 1$, so $o_g(hh') = 4$.

The following characterization of the elements of $\tau[g]$ of orbit number 1 further illustrates the connection with induced transformations.

Corollary 5.12. *Let (X, g) be a compact minimal system and let $h \in \tau[g]$. Then the following are equivalent:*

(i) $o(h) = 1$;

(ii) There is a nonempty clopen subset A of X, intersecting every infinite orbit of h, such that $h_A \in \{g_A, g_A^{-1}\}$.

Proof. We take the sign partition $h = h_p h_+ h_-$ of h. It is clear that h_p makes no contribution to the orbit number; it is also irrelevant for condition (ii). Thus we may assume $h_p = \mathrm{id}_X$.

Suppose $o(h) = 1$. Then $h \neq \mathrm{id}_X$; moreover, h acts minimally on its support, since otherwise $\langle h \rangle$ would have more than one infinite orbit on every $\langle g \rangle$-orbit. Thus $h = h_+$ or $h = h_-$. Suppose that $h = h_+$ and let A be set of points in $\mathrm{supp}(h)$ belonging to the strongly positive domain of h with respect to g. Then by Lemma 4.4 and Proposition 5.5: A is clopen and intersects every infinite orbit of h; we have $o(h) = o(h_A)$; and $h_A \in \tau_>[g]$. We see that h_A acts minimally on A, with exactly one infinite orbit on every $\langle g \rangle$-orbit on X; thus h_A acts transitively on $\langle g \rangle x \cap A$ for all $x \in X$. Since h_A acts only by nonnegative powers of g, the only possibility is that $h_A = g_A$. If instead $h = h_-$, we let A consist of those points in $\mathrm{supp}(h^{-1})$ in the strongly positive domain of h^{-1}, and the same argument as before shows that $h_A^{-1} = g_A$, so $h_A = g_A^{-1}$.

Conversely, suppose that there is a nonempty clopen subset A of X, intersecting every infinite orbit of h, such that $h_A \in \{g_A, g_A^{-1}\}$. Then $\langle h_A \rangle = \langle g_A \rangle$; thus by Proposition 5.5,

$$o(h) = o(h_A) = o(g_A) = o(g) = 1.$$

□

6. Normal forms for elements

6.1. **Periodic points and pure cycles.** Given an aperiodic homeomorphism g of a compact Hausdorff space X, there are many elements of $\tau[g]$ of finite order; however they are all of a special form, which refines the mininal-periodic partition in this case.

Definition 6.1. Let g be an aperiodic homeomorphism of the topological space X and let $n \geq 2$. A **pure n-cycle** of g is an element $h \in \tau[g]$ for which there exists a nonempty clopen subset A of X, called a **base** for h, with the following properties:

(i) We have $h^n = \mathrm{id}_X$, whereas the sets $A, hA, \dots, h^{n-1}A$ are pairwise disjoint;
(ii) h is supported on the union $\bigsqcup_{i=0}^{n-1} h^i A$;
(iii) For all $i \in \mathbf{Z}$, given $x, y \in h^i A$ then $c_{g,h}(x) = c_{g,h}(y)$.

Define the **signature** of a pure n-cycle h to be $[c_{g,h}(x), c_{g,h}(hx), \dots, c_{g,h}(h^{n-1}x)]$ for $x \in \mathrm{supp}(h)$, where the square brackets indicate a cyclically ordered n-tuple. Note that the signature is independent of the choice of x.

A **pure cycle** of g is a pure n-cycle for some $n \geq 2$. A **pure involution** is a pure 2-cycle.

We say $h \in \mathrm{Homeo}(X)$ is **pointwise periodic** if every orbit is finite, and **finite order** if $h^n = \mathrm{id}_X$ for some $n > 0$.

Proposition 6.2. *Let g be an aperiodic homeomorphism of the compact Hausdorff space X and let $h \in \tau[g]$ be pointwise periodic. Then h has finite order and there is a unique finite set S of pure cycles of g with the following properties:*

(i) Distinct elements of S have disjoint support (in particular, they commute);
(ii) No two elements of S have the same signature;
(iii) $h = \prod_{s \in S} s$.

Proof. By Lemma 2.1 we have a partition of X into clopen spaces $X_p(n)$, on which h has order n. Since X is compact, only finitely many of the spaces $X_p(n)$ are nonempty, so h has finite order.

Given $x \in X$, define the h-signature of x to be the linearly ordered tuple

$$(c_{g,h}(x), c_{g,h}(hx), \dots, c_{g,h}(h^{n-1}x)),$$

where n is the least positive integer such that $h^n x = x$. Define an equivalence relation E on X by setting $(x,y) \in E$ if x and y have the same h-signature. Since h has finite order and $c_{g,h}$ is continuous, we see that each of the E-classes is clopen, and there are finitely many classes. Define another equivalence relation E' by setting $(x,y) \in E'$ if the h-signature of y is a cyclic reordering of the h-signature of x. Then each E'-class is $\langle h \rangle$-invariant and partitioned into E-classes. Let $X_0, \dots, X_k$ be the set of E'-classes, where X_0 is the set of points with signature (0), and let $S = \{h_1, \dots, h_k\}$, where h_i is the restriction of h to X_i.

We now claim that each of the homeomorphisms h_i is a pure cycle. Fix $i \geq 1$ and let A be an E-class contained in X_i. We see that all orbits of h_i on X_i have the same length, say $n \geq 2$. From the definitions, it is easy to see that h_i is supported on $\bigcup_{j=0}^{n-1} h_i^j A$, that $h_i^n A = A$, and that for all $j \in \mathbf{Z}$, given $x, y \in h_i^j A$ then $c_{g,h}(x) = c_{g,h}(y)$.

Let $x \in A$. Since $|\langle h \rangle x| = n$ and g is aperiodic, given $j' \in \mathbf{Z}$ and $m \geq 0$ we see that

$$\sum_{j=0}^{m-1} c_{g,h}(h^{j'+j}x) = 0 \Leftrightarrow m \text{ is a multiple of } n.$$

In particular, it follows that if $n = dm$ for integers $m \geq 1$ and $d \geq 2$, then

$$\sum_{j'=0}^{d-1} f(j') = 0, \text{ where } f(j') := \sum_{j=0}^{m-1} c_{g,h}(h^{dj'+j}x),$$

but $f(j') \neq 0$ for all j'. Thus there is some j' such that $f(j') \neq f(j'+1)$, and hence some integer j such that $c_{g,h}(h^j x) \neq c_{g,h}(h^{j+m}x)$. This ensures that all n of the cyclic reorderings of the h-signature of x are distinct. Recalling the definitions of E and h_i, it follows that if j is not a multiple of n, then $h_i^j A$ is disjoint from A. This completes the proof that A is a base for h_i, and hence that h_i is a pure cycle.

We see from how the maps h_i were constructed that distinct elements of S have disjoint support and $h = \prod_{s \in S} s$. The signature of h_i is exactly

$$[c_{g,h}(x), c_{g,h}(hx), \dots, c_{g,h}(h^{n-1}x)],$$

where n is the length of the nontrivial orbits of h_i and x is any point in X_i; given how E' was defined, this ensures that no two elements of S have the same signature. For uniqueness, we see that the properties specified for S force the elements of S to be the restrictions of h to the E'-classes, except for the E'-class on which h acts trivially. □

6.2. **Normal forms with respect to induced transformations.** Let (X, g) be a compact minimal system. By Theorem 5.6, the system admits an index map, so Proposition 5.4 applies; using these results, we can now prove Theorem 1.8.

Proof of Theorem 1.8. Write I for the index map of (X, g). Recall that by Proposition 5.4(i), every element of $\tau_>[g]$ is a product of induced transformations.

(i) Let $h \in \tau_>[g]$. If $h = g_A$ for some nonempty clopen subset A of X, then $I(h) = 1$; since $I(k) > 0$ for all nontrivial $k \in \tau_>[g]$, it follows that h is irreducible. On the other hand if h is not an induced transformation of g, then h must be a product of at least two induced transformations, so h is reducible. Thus h is irreducible in $\tau_>[g]$ if and only if h is an induced transformation, as required.

(ii) By Proposition 5.4(i), every element has at least one suitable expression; it remains to show that the expression is unique.

Let $h = g_{A_m} \dots g_{A_2} g_{A_1} g^r$ and $h' = g_{B_n} \dots g_{B_2} g_{B_1} g^s$, where A_i and B_i are proper nonempty clopen subsets of X such that $A_{i+1} \subseteq A_i$ and $B_{i+1} \subseteq B_i$ for all i. Suppose that $h = h'$.

Without loss of generality, suppose that $r \geq s$. Then given $x \in X$ we have

$$c_{g,hg^{-s}}(x) = c_{g,hg^{-r}}(g^{r-s}x) + (r - s).$$

Our hypotheses ensure that $c_{g,hg^{-r}}(y) \geq 0$ for all $y \in X$, and also that there exists $x \in X$ such that $c_{g,hg^{-s}}(x) = 0$. This cannot be achieved if $r > s$, so we must have $r = s$, and hence

$$hg^{-r} = g_{A_m} \dots g_{A_2} g_{A_1} = g_{B_n} \dots g_{B_2} g_{B_1}.$$

From now on we can consider expressions for hg^{-r} rather than h, and so we may assume that $r = s = 0$.

We see that $m = I(h)$ and $n = I(h')$; since $h = h'$ we must have $m = n$. Now proceed by induction on m. An easy calculation shows that $c_{g,h}(x) > 0$ if and only if $x \in A_1$, and similarly $c_{g,h'}(x) > 0$ if and only if $x \in B_1$. Since $h = h'$ we must have $A_1 = B_1$. Thus

$$g_{A_m} \dots g_{A_2} = g_{B_n} \dots g_{B_2},$$

and by the inductive hypothesis, we have $A_i = B_i$ for $2 \leq i \leq m$. $\square$

Remark 6.3. Let (X, g) be a compact minimal system and let $h \in \tau[g]$. We can produce a canonical expression for h in terms of pure cycles and induced transformations of g in a way that retains the structure of the sign partition, as follows. Write

$$h = h_p(h^\sigma_> h_> (h^\sigma_>)^{-1})(h^\sigma_< h_< (h^\sigma_<)^{-1});$$

decompose h_p according to Proposition 6.2; and then decompose the strongly positive elements $h^\sigma_>$, $h_>$, $(h^\sigma_<)^{-1}$ and $h_<^{-1}$ according to Theorem 1.8.

One application of the characterization of irreducible elements is that if X is zero-dimensional, one can easily recover the conjugacy class of the compact minimal system (X, g) from the monoid structure of $\tau_>[g]$.

Proposition 6.4. *Let (X, g) be a compact zero-dimensional minimal system and consider $\tau_>[g]$ as a monoid. Let $\mathcal{A}$ be the set of irreducible elements of $\tau_>[g]$ together with the identity. Define the **support order** on $\mathcal{A}$ by setting $h_1 \leq_{\mathrm{supp}} h_2$ if*

$$\mathrm{C}_{\mathcal{A}}(h_2) \subseteq \mathrm{C}_{\mathcal{A}}(h_1) \cup \{h_2\}, \text{ where } \mathrm{C}_{\mathcal{A}}(h) := \{a \in \mathcal{A} \mid ah = ha\}.$$

Then $\mathcal{A}$ is a Boolean algebra with least element id_X and greatest element g, on which $\langle g \rangle$ acts by conjugation. Moreover, $(\mathcal{A}, \leq_{\mathrm{supp}})$ is $\langle g \rangle$-equivariantly isomorphic to the set $\mathcal{CO}(X)$ of compact open subspaces of X, ordered by inclusion.

Proof. By Theorem 1.8(i) we see that $\mathcal{A} = \{g_A \mid A \in \mathcal{CO}(X)\}$. By Lemma 2.7, given $A \in \mathcal{CO}(X)$ then $gg_A g^{-1} = g_{gA}$, so $\langle g \rangle$ acts on $\mathcal{A}$ by conjugation. It now suffices to show that $\leq_{\mathrm{supp}}$ corresponds to the inclusion order on clopen sets in the obvious manner: that is, given $A, B \in \mathcal{CO}(X)$, that $g_A \leq_{\mathrm{supp}} g_B$ if and only if $A \subseteq B$.

Suppose that $A \smallsetminus B$ is nonempty. Then there is a proper nonempty clopen subset A' of $A \smallsetminus B$. Since g_A acts minimally on A, it follows that $g_A A' \neq A'$; since $A' = \mathrm{supp}(g_{A'})$, this means that g_A and $g_{A'}$ do not commute. However, $g_{A'}$ does commute with g_B, since A' and B are disjoint. Thus $g_A \not\leq_{\mathrm{supp}} g_B$, proving that $g_A \leq_{\mathrm{supp}} g_B \Rightarrow A \subseteq B$.

Now suppose instead that $A \subseteq B$ and suppose $B' \in \mathcal{CO}(X) \smallsetminus \{B\}$ is such that g_B and $g_{B'}$ commute. Then g_B and $g_{B'}$ both preserve the set $B \cap B'$; since $B \cap B'$ cannot equal both B and B', by the minimality of g_B and $g_{B'}$ we must have $B \cap B' = \emptyset$. But then $A \cap B' = \emptyset$, so g_A and $g_{B'}$ also commute. Thus $g_A \leq_{\mathrm{supp}} g_B$, proving that $A \subseteq B \Rightarrow g_A \leq_{\mathrm{supp}} g_B$. □

By Stone duality, any compact zero-dimensional space can be recovered from its Boolean algebra of clopen subsets. The following corollary is thus immediate.

Corollary 6.5. *Let (X_1, g_1) and (X_2, g_2) be compact zero-dimensional minimal systems, and suppose that $\theta : \tau_>[g_1] \to \tau_>[g_2]$ is an isomorphism of monoids. Then $\theta(g_1) = g_2$ and there is a homeomorphism $\kappa : X_1 \to X_2$ such that $\kappa(hx) = \theta(h)(\kappa x)$ for all $x \in X_1$ and $h \in \tau_>[g_1]$.*

Another consequence of our normal form for strongly positive elements is that it leads to monoid presentations of $\tau_>[g]$ and $\tau[g]$ in terms of induced transformations.

Proposition 6.6. *Let (X, g) be a compact minimal system and let $\mathcal{CO}^*(X)$ be the set of nonempty clopen subsets of X. Then there is a unique binary operation $*$ on $\mathcal{CO}^*(X)$ such that for all $A, B \in \mathcal{CO}^*(X)$, we have*

$$g_A g_B = g_{A*B} g_{A \cup B}; \quad A * B \subseteq A \cup B.$$

With respect to the generating set $\{g_A \mid A \in \mathcal{CO}^*(X)\}$, $\tau_>[g]$ has the monoid presentation

$$\tau_>[g] = \langle \{g_A \mid A \in \mathcal{CO}^*(X)\} \mid g_A g_B = g_{A*B} g_{A\cup B} \rangle.$$

Similarly, with respect to the generating set $\{g_A \mid A \in \mathcal{CO}^*(X)\} \cup \{g^{-1}\}$, $\tau[g]$ has the monoid presentation (also a group presentation)

$$\tau[g] = \langle \{g_A \mid A \in \mathcal{CO}^*(X)\} \cup \{g^{-1}\} \mid g_A g_B = g_{A*B} g_{A\cup B}, g^{-1} g_A = g_{g^{-1}A} g^{-1}, g_X g^{-1} = 1 \rangle.$$

Proof. Note that $g_X = g$. Given $A, B \in \mathcal{CO}^*(X)$, we see that $h = g_A g_B$ is a strongly positive element such that $I(h) = 2$ and $\mathrm{supp}(h) = A \cup B$. The normal form of h is therefore $h = g_{A*B} g_{A\cup B}$, for some nonempty clopen set $A * B$ depending on A and B.

Let R_1 be the set of relations $g^{-1} g_A = g_{g^{-1}A} g^{-1}$ for $A \in \mathcal{CO}^*(X)$; let R_2 be the set of relations $g_A g_B = g_{A*B} g_{A\cup B}$ for $A, B \in \mathcal{CO}^*(X)$; let R_3 consist of the single relation $g_X g^{-1} = 1$; and let $R = R_1 \cup R_2 \cup R_3$. Now consider a word w in the alphabet $\{g_A \mid A \in \mathcal{CO}^*(X)\} \cup g^{-1}$:

(i) Using R_1 we can rearrange to obtain a word of the form $g_{A_n} \dots g_{A_2} g_{A_1} (g^{-1})^{r_1}$ for $A_i \in \mathcal{CO}^*(X)$ and $r_1 \geq 0$.
(ii) Using R_2, we rearrange the word to the form $g_{B_n} \dots g_{B_2} g_{B_1} (g^{-1})^{r_1}$, where now $B_i \in \mathcal{CO}^*(X)$ such that $B_{i+1} \subseteq B_i$ for all i.
(iii) Let m be maximal such that $0 \leq m \leq r_1$ and $X = B_1 = B_2 = \dots = B_m$. Then using R_3, we reduce to $g_{B_n} \dots g_{B_{m+2}} g_{B_{m+1}} k^{r_2}$ for some $r_2 \geq 0$, where k is either g^{-1} or g_X.

We declare the result of this process to be a reduced word.

By Theorem 1.8(ii), each element of $\tau[g]$ is represented by exactly one reduced word, so there are no monoid relations in $\tau[g]$ that are not already implied by R. Thus $\tau[g]$ has the monoid presentation

$$\tau[g] = \langle \{g_A \mid A \in \mathcal{CO}^*(X)\} \cup \{g^{-1}\} \mid R \rangle.$$

Since $\tau[g]$ is a group, this monoid presentation is also a group presentation for $\tau[g]$.

If w does not involve g^{-1}, then steps (i) and (iii) in the reduction process have no effect; moreover, we can represent every element of $\tau_>[g]$ with a reduced word that does not involve g^{-1}. Thus $\tau_>[g]$ has the monoid presentation

$$\tau_>[g] = \langle \{g_A \mid A \in \mathcal{CO}^*(X)\} \mid R_2 \rangle. \qquad \square$$

7. Strongly p.p.m. homeomorphisms

Fix a p.p.m. homeomorphism h of the compact Hausdorff space X, and consider the minimal homeomorphisms g such that $h \in \tau[g]$. We can take the orbit number $o_g(h)$ as a measure of the 'efficiency' with which g witnesses that h is p.p.m.; define $o_{\min}(h)$ to be the smallest value of $o_g(h)$, where g is a minimal homeomorphism of X such that $h \in \tau[g]$. It is clear that $o_{\min}(h) \geq m(h)$, where $m(h)$ is the number of distinct infinite minimal orbit closures of h on X; say that h is **strongly p.p.m.** if $o_{\min}(h) = m(h)$. In this section, we study the structure of strongly p.p.m. homeomorphisms.

We first note a decomposition for certain periodic automorphisms of compact zero-dimensional spaces.

Lemma 7.1. *Let X be a compact zero-dimensional Hausdorff space, let $h \in \mathrm{Homeo}(X)$ and write $X_p(n)$ for the set of points such that $|\langle h \rangle x| = n$. Let $n \geq 1$, and suppose that $X_p(n)$ is closed. Then there is a partition of $X_p(n)$ into clopen sets $X_p(n,0), \dots, X_p(n,n-1)$, such that $hX_p(n,i) = X_p(n,j)$ where $j = i+1 \mod n$.*

Proof. It is enough to consider the case $X = X_p(n)$; if $n = 1$ there is nothing to prove, so assume that $n > 1$. Since h has finite order on X, it also has finite order on $X \times X$; thus, given a clopen equivalence relation E on $X \times X$, there is a clopen $\langle h \rangle$-invariant equivalence relation given by $\bigcap_{k=0}^{n-1} h^k E$. Since X is zero-dimensional, we deduce that there is a base for a compatible uniformity on X that consists of $\langle h \rangle$-invariant clopen equivalence relations. For each $x \in X$, let U_x be a clopen neighbourhood of x such that the sets $U_x, hU_x, \dots, h^{n-1}U_x$ are pairwise disjoint; then by compactness we may choose a finite cover of X of the form $\{U_{x_1}, \dots, U_{x_m}\}$. There is then a clopen $\langle h \rangle$-invariant equivalence relation E on X such that each E-class is contained in U_{x_i} for some i. In particular, this ensures that all orbits of h on the quotient space X/E have size n; note also that the quotient space X/E is compact and discrete, hence finite, so there are only finitely many E-classes. Take $X_p(n,0)$ to be a union of the smallest possible number of E-blocks so that $\bigcup_{k \in \mathbf{Z}} h^k X_p(n,0) = X$, and set $X_p(n,i) = h^i X_p(n,0)$ for $1 \leq i < n$. We see that in fact X is the disjoint union of the sets $X_p(n,0), \dots, X_p(n,n-1)$, as required. $\square$

Using Lemma 7.1, it is easily seen that if X is the Cantor set and h is a homeomorphism of finite order such that $|\langle h \rangle x|$ depends continuously on $x \in X$, then h is strongly p.p.m.; in this case $o_{\min}(h) = 0$. These are the only p.p.m. homeomorphisms of X with $m(h) = 0$. As an example of a finite-order homeomorphism of the Cantor set that is *not* p.p.m., represent X as the set of infinite binary strings, let a be the map such that $a(0w) = 1w$ and $a(1w) = 0w$, and let h be the map such that $h(0w) = 0a(w)$ and $h(1w) = 1h(w)$. Then h has a single fixed point $111\dots$, so the set of fixed points of h is not clopen, and therefore h cannot be p.p.m.

The next proposition will allow us to reduce the study of strongly p.p.m. homeomorphisms to the aperiodic case: in particular, it implies that whenever $m(h) = 1$, then h is strongly p.p.m.

Proposition 7.2. *Let X be a generalized Cantor space and let $h \in \mathrm{Homeo}(X)$. Suppose that h admits a minimal-periodic partition, and let h_a be the restriction of h to the aperiodic part X_a. If $m(h) = 0$, then h is p.p.m. and $o_{\min}(h) = 0$. If $m(h) > 0$ then h is p.p.m. if and only if h_a is p.p.m.; if h is p.p.m., then $o_{\min}(h) = o_{\min}(h_a)$.*

Proof. Let X_p be the set of periodic points for h, and suppose either that X_a is empty or that $h_{X_a} \in \tau[g']$, where g' is a minimal homeomorphism of X_a. If X_a is empty, set $g' = \mathrm{id}_X$. Partition X_p into clopen sets $X_p(n,i)$ as in Lemma 7.1. Only finitely many of these sets are nonempty; let us say that

$$\{X_p(n,i) \mid 0 \leq i < n, X_p(n,i) \neq \emptyset\} = \{Y_1, \ldots, Y_l\}.$$

We arrange the sets Y_j so that for $j < j'$, if $Y_j = X_p(n,i)$ and $Y_{j'} = X_p(n',i')$, then either $n < n'$ or $n = n'$ and $i < i'$.

Since X is a generalized Cantor space, the sets $Y_1, \ldots, Y_l$ are all homeomorphic to one another; if X_a is nonempty then X_a is homeomorphic to Y_1. Set $\epsilon = 1$ if X_a is empty; if X_a is nonempty, write $X_a = Y_0$ and set $\epsilon = 0$. Choose homeomorphisms $t_j : Y_j \to Y_{j+1}$ for $\epsilon \leq j < l$. In the case that $Y_j = X_p(n,i)$ and $Y_{j+1} = X_p(n,i+1)$ for some n and i, then we define t_j by setting $t_j x = hx$ for all $x \in Y_j$; otherwise choose an arbitrary homeomorphism.

Now define a homeomorphism g on X, by setting gx as follows:

If $x \in Y_j$ for $\epsilon \leq j < l$, set $gx = t_j x$.

If $x \in Y_l$, set $gx = g' t_\epsilon^{-1} t_{\epsilon+1}^{-1} \ldots t_{l-1}^{-1} x$.

In the case that X_a is empty, it is clear that we obtain $h \in \tau[g]$ with $o_g(h) = 0$.

If X_a is nonempty, observe that for all $x \in X_a$, we have $g^{l+1}x = g'x$; thus each forward g-orbit has dense intersection with X_a, and the action of h on X_a is given locally by powers of g. The other powers of g ensure that g acts minimally on the whole of X. When $x \in X_p(n,i)$ for $0 \leq i < n-1$, then $hx = gx$; when $x \in X_p(n,n-1)$, then $hx = g^{1-n}x$. Thus $h \in \tau[g]$, showing that h is p.p.m. The construction ensures that $o_g(h) = o_{g'}(h_a)$, and hence $o_{\min}(h) \leq o_{\min}(h_a)$.

Conversely, suppose that X_a is nonempty and h is p.p.m., say $h \in \tau[g]$ where $g \in \mathrm{Homeo}(X)$ is minimal. Then g_{X_a} acts minimally on X_a and we have $h_{X_a} \in \tau[g_{X_a}]$, so the restriction of h to X_a is p.p.m. We see that $o_g(h) = o_{g_{X_a}}(h_a)$, so $o_{\min}(h_a) \leq o_{\min}(h)$. □

Let (X_1, g_1) and (X_2, g_2) be compact minimal systems. A **Kakutani equivalence** of (g_1, g_2) is a homeomorphism $\kappa : Y_1 \to Y_2$, where Y_i is a nonempty clopen subset of X_i, such that $\kappa \circ (g_1)_{Y_1} = (g_2)_{Y_2} \circ \kappa$. We say that g_1 and g_2 are **Kakutani equivalent** if a Kakutani equivalence exists, and **flip Kakutani equivalent** if g_1 is Kakutani equivalent to g_2 or g_2^{-1}. This concept was introduced in the ergodic setting by S. Kakutani, then translated to topological dynamics by later authors (see [9]).

Lemma 7.3. *Kakutani equivalence and flip Kakutani equivalence are equivalence relations.*

Proof. It is clear that Kakutani equivalence is reflexive and symmetric. To show transitivity, let (X_i, g_i) be a compact minimal system for $1 \leq i \leq 3$ and suppose we have Kakutani equivalences $\kappa_{12} : Y_1 \to Y_2$ and $\kappa_{23} : Z_2 \to Z_3$, where Y_i and Z_i are nonempty clopen subsets of X_i. Since g_2 is minimal, there is some nonempty clopen subset W of Y_2 and $t \geq 0$ such that $g_2^t W \subseteq Z_2$. Note that the restriction of κ_{12} to a homeomorphism from $Y_1' = \kappa^{-1}(W)$ to $Y_2' = W$ is also a Kakutani equivalence of (g_1, g_2), since $((g_i)_{Y_i})_{Y_i'} = (g_i)_{Y_i'}$. Thus we may assume without loss of generality that $Y_2 = W$.

Similarly, we may assume without loss of generality that $Z_2 = g_2^t Y_2$. In this case, by Lemma 2.7 we see that $g_2^t (g_2)_{Y_2} = (g_2)_{Z_2} g_2^t$.

We now obtain a homeomorphism κ_{13} from Y_1 to Z_3 by setting $\kappa_{13}x = \kappa_{23} g_2^t \kappa_{12} x$. Let $x \in Y_1$. Then

$$\begin{aligned}\kappa_{13}(g_1)_{Y_1} x &= \kappa_{23} g_2^t \kappa_{12} (g_1)_{Y_1} x = \kappa_{23} g_2^t (g_2)_{Y_2} \kappa_{12} x \\ &= \kappa_{23}(g_2)_{Z_2} g_2^t \kappa_{12} x = (g_3)_{Z_3} \kappa_{23} g_2^t \kappa_{12} x = (g_3)_{Z_3} \kappa_{13} x.\end{aligned}$$

Thus κ_{13} is a Kakutani equivalence of (g_1, g_3). This proves that Kakutani equivalence is transitive and hence it is an equivalence relation. It is then clear that flip Kakutani equivalance is also an equivalence relation. □

A minimal homeomorphism is clearly Kakutani equivalent to its induced transformations on nonempty clopen sets. In particular, if (X, g) is a compact minimal system and $\{X_1, \dots, X_n\}$ is a partition of X into clopen sets, then the compact minimal systems (X_i, g_{X_i}) lie in a single Kakutani equivalence class. In fact, all finite tuples of Kakutani equivalent systems arise in this way.

Proposition 7.4. *Let n be a natural number and let $((X_i, g_i))_{1 \le i \le n}$ be an n-tuple of Kakutani equivalent compact minimal systems. Then there is a minimal homeomorphism g of the disjoint union $X = \bigsqcup_{i=1}^n X_i$, such that $g_{X_i} = g_i$ for $1 \le i \le n$.*

Proof. Suppose $n \ge 3$ and that the proposition is true for all smaller choices of n. Then there is a minimal homeomorphism g' of $X' = \bigsqcup_{i=1}^{n-1} X_i$ such that $g'_{X_i} = g_i$ for $1 \le i \le n-1$. Now g' is Kakutani equivalent to g_n, so there is a minimal homeomorphism g of $X = X' \sqcup X_n$ such that $g_{X'} = g'$ and $g_{X_n} = g_n$. We then see that $g_{X_i} = g_i$ for $1 \le i \le n-1$. Thus it suffices to prove the result for $n \le 2$. Since the case $n = 1$ is trivial, we assume $n = 2$.

Choose a Kakutani equivalence $\kappa : Y_1 \to Y_2$ of (g_1, g_2), where Y_i is a nonempty clopen subset of X_i. By restricting κ, we may ensure that $g_i Y_i$ is disjoint from Y_i. Now let $X = X_1 \sqcup X_2$ and define $g : X \to X$ by setting

$$gx = \begin{cases} g_2 \kappa x & \text{if } x \in Y_1 \\ g_1 (g_1)_{Y_1}^{-1} \kappa^{-1} x & \text{if } x \in Y_2 \\ g_1 x & \text{if } x \in X_1 \smallsetminus Y_1 \\ g_2 x & \text{if } x \in X_2 \smallsetminus Y_2 \end{cases}.$$

We now compare g_{X_i} with g_i. If $x \in X_i \smallsetminus Y_i$, then certainly $g_{X_i} x = g_i x$. If $x \in Y_{3-i}$, then the sequence $gx, g^2x, \dots$ first passes through $g_i Y_i$, then follows the forward g_i-orbit until it reaches Y_i, then at the next step moves to $g_{3-i} Y_{3-i}$. The first point on the forward g_i-orbit in Y_i after visiting $g_i Y_i$ is given by applying $(g_i)_{Y_i} g_i^{-1}$.

The result is as follows: if $x \in Y_1$, then

$$\begin{aligned} g_{X_1} x &= (g_1 (g_1)_{Y_1}^{-1} \kappa^{-1})((g_2)_{Y_2} g_2^{-1}) g_2 \kappa x = g_1 (g_1)_{Y_1}^{-1} \kappa^{-1} (g_2)_{Y_2} \kappa x = \\ &= g_1 (g_1)_{Y_1}^{-1} \kappa^{-1} \kappa (g_1)_{Y_1} x = g_1 x. \end{aligned}$$

If $x \in Y_2$, then

$$g_{X_2} x = (g_2 \kappa)((g_1)_{Y_1} g_1^{-1})(g_1 (g_1)_{Y_1}^{-1} \kappa^{-1}) x = g_2 x.$$

Thus for all $1 \le i \le 2$ and all $x \in X_i$, we have $g_{X_i} x = g_i x$, proving that $g_{X_i} = g_i$. We see from the construction that g is bijective and is a local homeomorphism, so g is a homeomorphism. By the minimality of g_{X_i} on X_i, any nonempty closed g-invariant set contains X_1 or X_2; since $gY_1 = Y_2$, it follows that g is in fact minimal. □

We can now characterize the strongly p.p.m. property in terms of flip Kakutani equivalence.

Proposition 7.5. *Let X be a compact Hausdorff space and let $h \in \mathrm{Homeo}(X)$. Suppose that h admits a minimal-periodic partition, and that either h is aperiodic or X is a generalized Cantor space. Then h is strongly p.p.m. if and only if either $m(h) = 0$, or $m(h) \geq 1$ and the spaces (Y, h) for Y an infinite orbit closure of h all lie in a single flip Kakutani equivalence class.*

Proof. Let $m = m(h)$. Suppose for the moment that h is aperiodic, and write $X_1, \dots, X_m$ for the distinct infinite orbit closures of h.

If the systems (X_i, h) are all flip Kakutani equivalent, choose a homeomorphism h_i on X_i, so that the systems (X_i, h_i) are all Kakutani equivalent and h acts as either h_i or h_i^{-1} on X_i. By Proposition 7.4 there is a minimal homeomorphism g of X such that $h_i = g_{X_i}$; in particular, we can regard h_i as an element of $\tau[g]$ with $o_g(h_i) = 1$. It is then clear that $h \in \tau[g]$, with $o_g(h) = \sum_{i=1}^m o_g(h_i) = m$.

Conversely, suppose that h is strongly p.p.m., that is, there is a minimal homeomorphism g such that $h \in \tau[g]$ with $o_g(h) = m$. Let h_i be the restriction of h to X_i. Then $o_g(h_i) \geq 1$ for each i, since there exist infinite orbits of h_i, and

$$m = o_g(h) = \sum_{i=1}^{m} o_g(h_i).$$

We therefore have $o_g(h_i) = 1$ for all i. By Corollary 5.12, there is an induced transformation of h_i that is equal to either an induced transformation of g or g^{-1}; in particular, (X_i, h) is flip Kakutani equivalent to (X, g).

In the remaining case, h is not aperiodic and X is a generalized Cantor space. If $m = 0$ then h is strongly p.p.m. by Proposition 7.2. If $m \geq 1$, Proposition 7.2 shows that h is strongly p.p.m. if and only if (X_a, h) is strongly p.p.m.; thus we reduce to the aperiodic case, with the desired conclusion. □

Corollary 7.6. *Let X be a generalized Cantor space and let $h \in \mathrm{Homeo}(X)$. If $m(h) \leq 1$, then h is strongly p.p.m.*

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