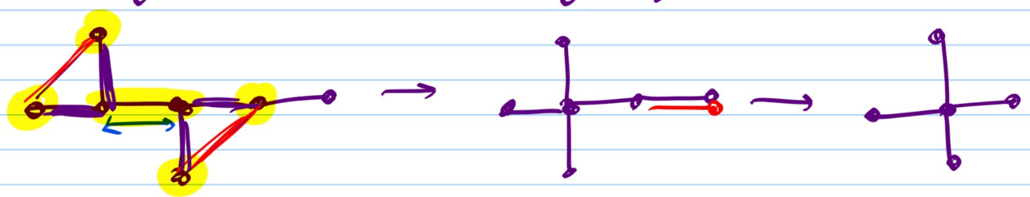
A new characterisation of virt. free groups.

§ Introduction

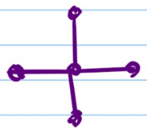
Δ - finite graph , Γ - graph

Δ is a *minor* of Γ if can obtain Δ from Γ by contracting edges and deleting edges and vertices:

e.g. Γ



Δ



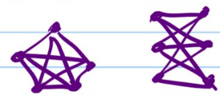
$\Delta < \Gamma$

$V(\Delta) = \{v_1, v_2, \ldots, v_n\}$ vertices.

Def. Δ is a minor of Γ if $\exists$ finite connected disjoint subsets $V_1, V_2, \ldots, V_n$ of $V(\Gamma)$ st. $\exists$ edge between some vertex of V_i and V_j if $(v_i, v_j) \in E(\Delta)$.

V_i - "branch sets" of Δ in Γ.

Th'm (Kuratowski, 1930)



A graph Γ is *planar* $\iff$ $K_5, K_{3,3} \not< \Gamma$

Th'm (Robertson-Seymour, 1983-2004)

Any property that passes to minors can be characterized by a finite list of "forbidden minors".

Conjecture (Hadwiger, 1943)

$$K_m \not< \Gamma \implies \chi(\Gamma) < m$$

Def. Γ is *minor-excluded* if $\exists$ finite graph Δ st. $\Delta \not< \Gamma$. ($\iff \exists m$ st. $K_m \not< \Gamma$).

Th'm (Ostrovskii-Rosenthal, 2014)

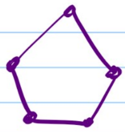
Γ loc. finite, connected graph, $K_m \not< \Gamma \implies \operatorname{asdim} \Gamma \leq 4^m - 1$

§ What about groups?

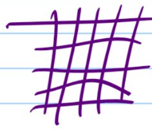
Th'm (Maschke, 1896)

Every finite *planar* group is a group of isometries of S^2.

admitting a gen set wrt which $Cay(G)$ is planar.

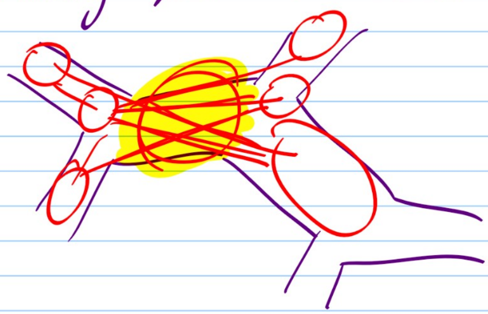
eg $\mathbb{Z}_5$   gen. set is important!

<u>Th'm</u> (Arzhantseva–Cherix, 2004)
Planarity is preserved by free products.



<u>Th'm</u> (O.–R.)
Minor excl. depends on the gen set. (eg. $\mathbb{Z}^2$)

<u>Th'm</u> (O.–R.)
Virtually free $\Rightarrow$ minor-excl. wrt. all finite gen sets.



<u>Question</u>: Is the converse true?

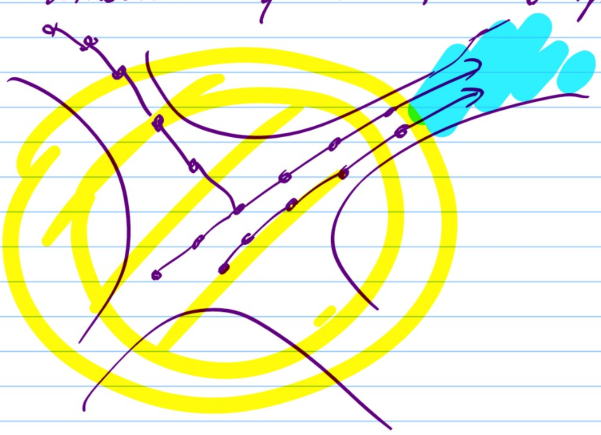
Virtually free $\Longleftrightarrow$ quasi-isom. to tree
$\Longleftrightarrow$ fin. pres. and asdim $= 1$.
$\Longleftrightarrow$ context-free word problem ...

- many use Dunwoody's Accessibility result. (cf Antolin)

<u>§ Ends and Accessibility.</u>

ray: 

Two rays in Γ are equivalent if they stay in the same connected of $\Gamma \setminus F$ $\forall$ finite $F \subset \Gamma$.



The equiv. classes are the <u>ends</u> of Γ.
The no. of ends is Q-I. invariant for groups. $=: \underline{e(G)}$.

<u>Th'm</u> (Hopf, 1943)

- $e(G) \in \{0, 1, 2, \infty\}$

- $e(G) = 0 \iff G$ finite
- $e(G) = 2 \iff G$ is virt. $\mathbb{Z}$.

<u>Th'm</u> (Stallings)

- $e(G) = \infty \iff G = A *_C B$ or $A *_C$ (C finite, $|A/C| \geq 3$, $|B/C| \geq 2$).

$\rightsquigarrow \quad e(G) > 1 \iff G = \underline{A *_C B}$ or $A *_C$.

$$\begin{array}{cc} \wedge & \wedge \end{array}$$

$$(\underline{D} *_E \underline{F}) *_C (- -)$$

G <u>accessible</u> if the process of decomposition terminates in a finite no. of steps.

<u>Th'm</u> (Dunwoody)

G finitely presented $\Rightarrow G$ accessible.

<u>Th'm</u> (O.-R.)

- $e(G) = 0 \Rightarrow \mathrm{Cay}(G,S)$ minor excl. $\forall S$
- $e(G) = 1 \Rightarrow \exists S$ st. $\mathrm{Cay}(G,S)$ is not minor excl. $(\star)$
- $e(G) = 2 \Rightarrow \mathrm{Cay}(G,S)$ minor. excl. $\forall S$.

<u>Q:</u> $e(G) = \infty$ ...?

<u>§ Characterising virt. free groups.</u>

<u>Th'm</u> (K., 2020)

If G is a f.g. group that is minor-excl. wrt any gen. set, then G is virt. free.

<u>Prop.</u> True if G is accessible

<u>Proof.</u>

G accessible $\Rightarrow G = \pi_1$ (finite graph of groups with finite edge groups and vertex groups with $e \leq 1$).

By $(\star)$, we cannot have subgroups with $e = 1$.

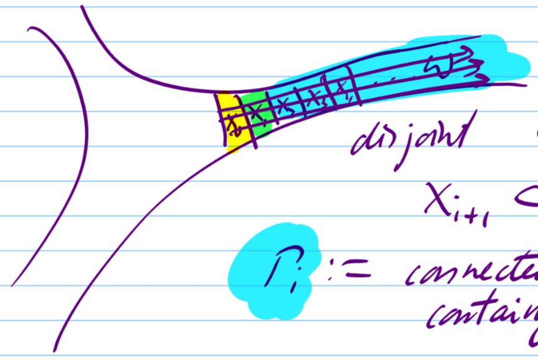
So all vertex groups must be finite, and so G virt. free (by Karass–Pietrowski–Solitar, 1973).

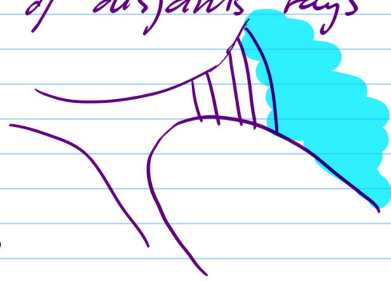
<u>Th'm</u> G minor excl. $\underline{\forall S} \Rightarrow G$ accessible.

<u>Sketch:</u> Thomassen–Woess (1991)

Set $\langle S \rangle = G$

G is not accessible $\iff$ $\forall m > 0$ $\mathrm{Cay}(G,S)$ has a "thin" end st. the max. no. of disjoint rays in this end is $\geq m$.



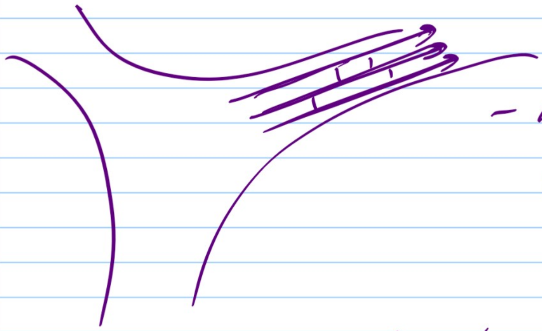


disjoint st. $\forall i \geq 0$

$X_{i+1} \subset \Gamma_i$, $\Gamma_{i+1} \subset \Gamma_i$

$\Gamma_i :=$ connected component of $\mathrm{Cay}(G,S) \setminus X_i$ containing ω.

ω thin if $\exists k$ st. $|X_i| = k$ $\forall i$.



- we use m rays to build a K_m minor in $\mathrm{Cay}(G, S \cup S^2 \cup S^3)$

- doesn't use Dunwoody
- $\exists$ G, S st $e(G) = \infty$ and $\mathrm{Cay}(G,S)$ not minor excl.

<u>Q:</u> Characterise the groups admitting $\geq$ one gen. set wrt. which $\mathrm{Cay}(G)$ is minor excl.