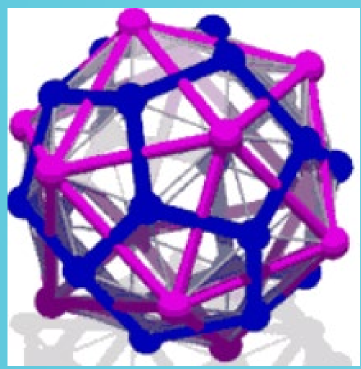
ICOSAHEDRON SYMMETRY GROUP AND ROTATION MATRICES

Introduction

A regular convex icosahedron has 20 faces, 30 edges and 12 vertices. It is duel to the ***dodecahedron***.

The elements of the symmetry group ofthe icosahedron:



Rotation symmetry

i. 4 rotations by multiples of $2\pi/5$ ($2\pi/5$, $4\pi/5$, $6\pi/5$, $8\pi/5$) about centre of 6 pairs of opposite faces, $4 \times 6 = 24$ elements

ii. 2 rotation by multiples of $2\pi/3$ ($2\pi/3$, $4\pi/3$) about the centre of 10 pairs of opposite faces, $2 \times 10 = 20$ elements

iii. 1 rotation by π about 15 pairs of opposite edges, $1\times15 = 15$ elements

In total, the rotational symmetry group has 24 + 20 + 15 = 59 plus 1 neutral element. (60 elements)

Mirror reflection:

Symmetrical along the plate orthogonal to the 15 pairs of opposite edge passing through the centre of icosahedron, 60 elements. (Isomorphic to cyclic group Z_2)

Therefore, the symmetry group of icosahedron contains **120 elements**, in other words, its order is 120.

Theorem: the rotational symmetry group of icosahedron is isomorphic to A_5 (A stands for alternating group) $A_5 \rightarrow SO_3(\mathbb{R})$

Representation theory

A representation of a group G is a homomorphism from G into the general linear group of a vector space V. **G → GL(V)**

For example, mapping S_3 to $GL(\mathbb{R}^3)$

S_3 is the symmetry group of the set {1, 2, 3}, in other words, the permutations of the set {1, 2, 3}. We can see that the permutations (1,2), (2,3) and (1,3) generate the six elements of S_3.

In $\mathbb{R}^3$, there are 3 axis (x, y, z) which can represent number 1, 2 and 3.

Use matrices multiplication, we can switch 3 axis around as follows:

Firstly, I have identity matrix as neural element $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Then, switch x and y by using $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ x \\ z \end{bmatrix}$

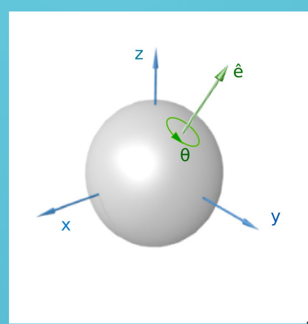
Similarly, switch y and z by using $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, and switch x and z by using $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

This is a good example of an isomorphism between S3 and a group of matrices.

Euler's rotation theorem and Euler axis and angle

- The theorem states that 'in 3D space, any displacement of a rigid body such that a point on the rigid body remains fixed, is equivalent to a single rotation.'
- The rotation axis is called a Euler axis, its product by rotation angle is known as an axis-angle.

This provide knowledge how to approach rotation group SO_3 and rotation matrix.

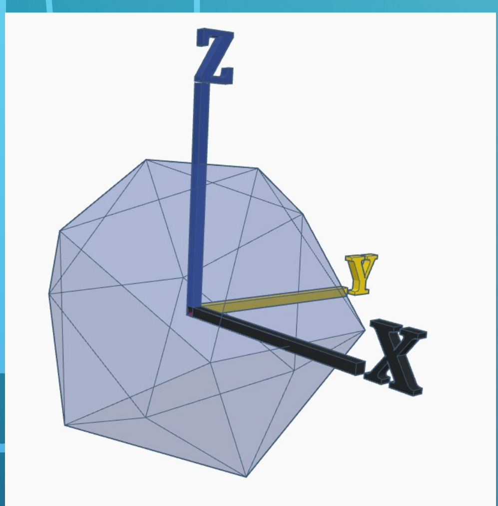


A rotation matrix can be calculated by knowing its Euler axis and angle

$R = (\cos\theta)\, I + (\sin\theta)[u]_x + (1-\cos\theta)\,(u \otimes u)$

(Note: u (u_x, u_y, u_z) is the unit vector represent axis and θ is the angle of rotation. $[u]_x$ is cross product matrix of u. $u \otimes u$ is the out product.)

Here, we combine group theory, Euler's rotation theorem and representation theory to create basic rotation matrices



Rotation by $2\pi/5$ around axis (0, 1, φ)

$$R_{2\pi/5} = \begin{bmatrix} \cos\left(\frac{2\pi}{5}\right) & -\frac{\varphi}{\sqrt{1+\varphi^2}}\sin\left(\frac{2\pi}{5}\right) & \frac{1}{\sqrt{1+\varphi^2}}\sin\left(\frac{2\pi}{5}\right) \\ \frac{\varphi}{\sqrt{1+\varphi^2}}\sin\left(\frac{2\pi}{5}\right) & \cos\left(\frac{2\pi}{5}\right)+\frac{1}{1+\varphi^2}\left(1-\cos\left(\frac{2\pi}{5}\right)\right) & \frac{\varphi}{1+\varphi^2}\left(1-\cos\left(\frac{2\pi}{5}\right)\right) \\ -\frac{1}{\sqrt{1+\varphi^2}}\sin\left(\frac{2\pi}{5}\right) & \frac{\varphi}{1+\varphi^2}\left(1-\cos\left(\frac{2\pi}{5}\right)\right) & \frac{\varphi^2}{1+\varphi^2}\left(1-\cos\left(\frac{2\pi}{5}\right)\right)+\cos\left(\frac{2\pi}{5}\right) \end{bmatrix}$$

$$R_{2\pi/5} = \begin{bmatrix} \frac{\sqrt{5}-1}{4} & -\left(\frac{\sqrt{5}+1}{4}\right) & \frac{1}{2} \\ \frac{\sqrt{5}+1}{4} & \frac{1}{2} & \frac{\sqrt{5}-1}{4} \\ -\frac{1}{2} & \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} \end{bmatrix}$$

Rotation by $2\pi/3$ around axis (1, 1, 1)

$$R_{2\pi/3} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Rotation by π around Z axis

$$R_{\pi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Mirror reflection along Z, X axis

$$F_Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Mirror reflection along Z axis

$(0, \pm1, \pm\varphi)$ $(0, \pm\varphi, \pm1)$

$(\pm1, \pm\varphi, 0)$ $(\pm\varphi, \pm1, 0)$

$(\pm\varphi, 0, \pm1)$ $(\pm1, 0, \pm\varphi)$

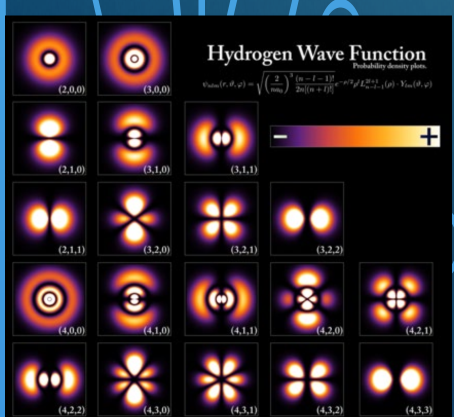
Greek letter φ stand for the golden ratio $\frac{1+\sqrt{5}}{2} \approx 1.618$

The neutral element is 3×3 identity matrix I

Elements generated from $R_{2\pi/5}$: $(R_{2\pi/5})^2$, $(R_{2\pi/5})^3$, $(R_{2\pi/5})^4$; Elements generated from $R_{2\pi/3}$: $(R_{2\pi/3})^2$

Order of the elements: $|R_{2\pi/5}| = 5$; $|(R_{2\pi/5})^2| = 5$; $|(R_{2\pi/5})^3| = 5$; $|(R_{2\pi/5})^4| = 5$; $|R_{2\pi/3}| = 3$; $|(R_{2\pi/3})^2| = 3$; $|R_\pi| = 2$; $|F_z| = 2$

Inverse: $(R_{2\pi/5})^{-1} = (R_{2\pi/5})^4$; $[(R_{2\pi/5})^2]^{-1} = (R_{2\pi/5})^3$. $(R_{2\pi/3})^{-1} = (R_{2\pi/3})^2$. $(R_\pi)^{-1} = R_\pi$. $(F_z)^{-1} = F_z$.



Connection between group theory and quantum mechanics

Representation theory is fundamental to quantum mechanics. It allows physicists to gain information about quantum mechanical state spaces when a group acts on a system. For example, quantum particles such as hydrogen atom can be represented by a complex-valued "wavefunction" – Lie Group. It acts on such wavefunctions by pointwise phase transformations of the value of the function. This allows us to understand and describe how particles interact with electromagnetic fields. Otherwise, it would be very hard for scientists to grasp the unworldly strange behaviour of the quantum particles.