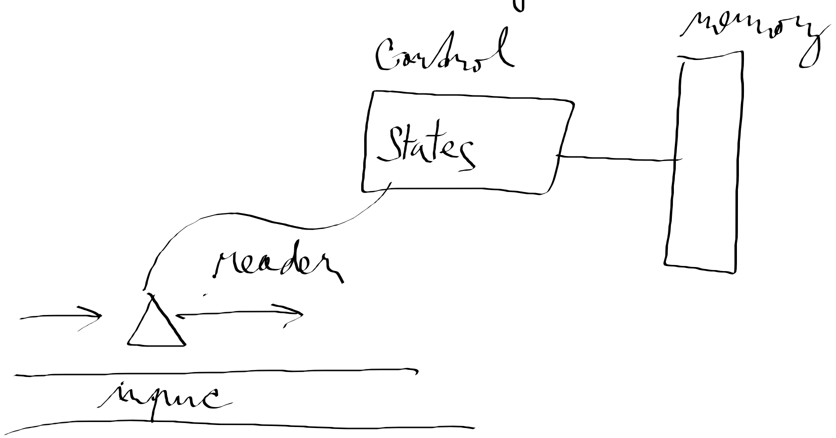
# Inverse limits of finite state automata & profinite words

$|X| < \infty$ $\quad L \subseteq X^*$ a language

$L$ is Regular/Rational if $L$ is accepted by FSA

Context-free PDA




FSA - no memory

PDA - push-down stack

$G = \langle X \mid R \rangle \qquad (X = X^{-1})$

$WP(G) = \{ w \in X^* \mid w =_G 1 \}$

1) Anisimov $WP(G)$ is Regular $\iff |G| < \infty$

2) Muller–Schupp : $WP(G)$ is CF $\iff$ $G$ is virtually free

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What about T. D. (L.) C groups?

## definition of X-FSA

An X-FSA is a tuple $M = (Q, q_0, A, \delta)$

$Q$ - finite set of states

$q_0 \in Q$ initial state

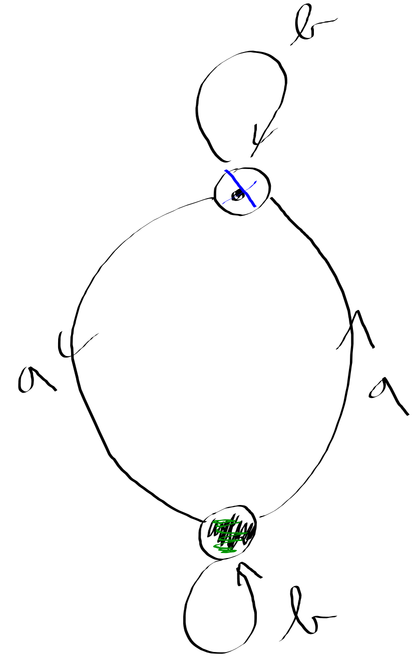
$A \subseteq Q$ accepting states

$\delta \subseteq Q \times X \times Q$ transition relation

For $w = x_1 \cdots x_n \in X^*$, $q \in Q$ "$w$ takes $q$ to $q'$"

if $\exists\, q_1, \dots, q_{n-1} \in Q$ : $(q, x_1, q_1) \in \delta$, $(q_{n-1}, x_n, q') \in \delta$

$(q_{i-1}, x_i, q_i) \in \delta$



$w(q) = \{ q' \in Q \mid \text{"}w \text{ takes } q \text{ to } q'\text{"} \} \subseteq Q$

$M$ accepts $w$ if $w(q_0) \cap A \neq \emptyset$

$L(M) = \{ w \in X^* \mid M \text{ accepts } w \}$

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$M = (Q, q_0, A, \delta)$, $M' = (Q', q_0', A', \delta')$

$f : Q \longrightarrow Q'$ is a X-FSA morphism if

$f(q_0) = q_0'$

$f(A) \subseteq A'$ $\leftarrow$ $\forall x \in X, q_1, q_2 \in Q$

$(q_1, x, q_2) \in \delta \implies (f(q_1), x, f(q_2)) \in \delta'$

$\Rightarrow L(M) \subseteq L(M')$

$I$ is poset , $M = (M_i)_{i \in I}$ $\quad \forall i \geq j : \exists f_{i,j} : M_i \to M_j$

$\forall i \geq j \geq k : \quad f_{i,k} = f_{j,k} \circ f_{i,j}$

$\widehat{M}_I = \varprojlim_{i \in I} M_i$

$$\begin{tikzcd} & M_i \arrow[dl] \arrow[dr] & \\ M_j \arrow[rr] & & M_k \end{tikzcd}$$

$\widehat{M}_I$ has set of states that is Stone space (profinite)

$f : M \longrightarrow M'$ , $w, w' \in X^*$

$(w, w')$ is $f$-compatible if $\quad f(w(q_0)) = w'(q_0')$

$$\widehat{X^*_I} = \{ (w_i)_I \mid (w_i, w_j) \text{ is } f_{i,j}\text{-compatible} \}$$

$\quad \| $

$\underline{\text{profinite words}}$

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if $\underset{\langle X \rangle}{\overset{}{G}} = \varprojlim G_i \qquad M_i \simeq \mathcal{C}ay(G_i, \pi_i(X))$

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if $M$ is a $X$-FSA s.t. $L(M) = WP(G)$

then there is $f : M \to M'$ s.t. $L(M) = L(M')$

$M' \simeq \mathcal{C}ay(G, X)$