Topological full groups.

Etale groupoids

A groupoid $\mathcal{G}$ is a set with a multiplication $\mathcal{G}\times\mathcal{G}\supseteq\mathcal{G}^{(2)}\longrightarrow\mathcal{G}$ $(g,h)\mapsto gh$ & an inversion $\mathcal{G}\to\mathcal{G}$, $g\mapsto g^{-1}$ satisfying

- $(gh)k = g(hk)$ whenever one & hence both sides are defined
- $g^{-1}gh=h$ & $ghh^{-1}=g$ for all $(g,h)\in\mathcal{G}^{(2)}$.

As a consequence

- $(g^{-1})^{-1} = g$.
- $(gh)^{-1} = h^{-1}g^{-1}$

Define source & range maps s, r

$$s(g) = g^{-1}g \quad \& \quad r(g) = gg^{-1}$$

Then $g = r(g)\, g\, s(g)$

$gh = g\, s(g)\, r(h)\, h$ so gh makes sense iff $s(g) = r(h)$.

$$\begin{tikzcd} r(g) & s(g)=r(h) \arrow[l, bend right, "g"'] & s(h) \arrow[l, bend right, "h"'] \arrow[ll, bend left, "gh"] \end{tikzcd}$$

Write $\mathcal{G}^{(0)} = s(\mathcal{G}) = r(\mathcal{G})$ for the <u>unit space</u>

A <u>topological groupoid</u> is a groupoid $\mathcal{G}$ with a topology such that $\mathcal{G}^{(2)} \to \mathcal{G}$ & $\mathcal{G} \to \mathcal{G}$ are continuous. A topological groupoid $\mathcal{G}$ is étale if it is locally compact, Hausdorff, & r, s are local homeomorphisms.

$$\begin{array}{ll} \mathcal{G}^{(2)} \to \mathcal{G} & \mathcal{G} \to \mathcal{G} \\ (g,h) \mapsto gh & g \mapsto g^{-1} \end{array}$$

Examples

G group
$G^{(0)} = \{e\}$

G étale $\iff$ discrete

$G \ltimes X$

X locally cpt Hdf.
$X^{(0)} = X$

G ———————————————— X

$(G_x)_{x \in X}$, G_x group

Equivalence relation $\sim$ on X

$\mathcal{G} = \{(x,y) : x,y \in X,\ x \sim y\}$

$(x,y)(y,z) = (x,z)$ $\quad \mathcal{G}^{(0)} \cong X$

$(x,y)^{-1} = (y,x)$

If $G \times X \longrightarrow X$ is a group action

Define the transformation groupoid

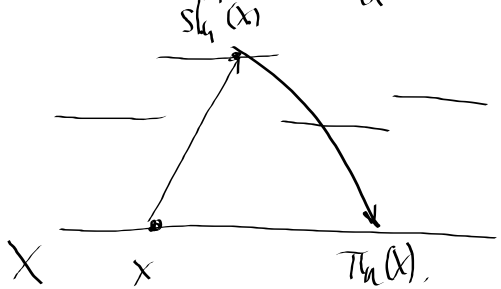
$$G \ltimes X = \{(g, x) : g \in G, x \in X\}$$

$$\begin{tikzcd}
g \cdot x & x = h \cdot y \arrow[l, bend right, "{(g,x)}"'] & y \arrow[l, bend right, "{(h,y)}"'] \arrow[ll, bend left, "{(gh,y)}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
gx \arrow[r, bend right, "{(g^{-1}, gx) = (g,x)^{-1}}"'] & x \arrow[l, bend right, "{(g,x)}"']
\end{tikzcd}$$

$$(G \ltimes X)^{(0)} = \{(e, x) : x \in X\} \simeq X.$$

fix a groupoid G with $G^{(0)} = X$ - the Cantor set.
We say a clopen subset $U \subseteq G$ is a <u>bisection</u> if
both $r|_U$ & $s|_U$ are homeomorphisms
We. call a bisection U <u>global</u> if $r(U) = s(U) = X$
A global bisection U determines a homeomorphism

$$\pi_U = r \circ s|_U^{-1} \in \mathrm{Homeo}(X)$$



We have

$\pi_{\mathcal{U}}^{-1} = \pi_{\mathcal{U}^{-1}}$ where $\mathcal{U}^{-1} = \{g^{-1} : g \in \mathcal{U}\}$ because

$s(g^{-1}) = r(g)$, $r(g^{-1}) = s(g)$.

$\pi_{\mathcal{U}} \circ \pi_{\mathcal{V}} = \pi_{\mathcal{U}\mathcal{V}}$ where $\mathcal{U}\mathcal{V} = \{gh : g \in \mathcal{U}, h \in \mathcal{V}\}$

$\pi_{\mathcal{G}^{(0)}} = id_X$

The topological full group

$[[\mathcal{G}]] = \{\pi_{\mathcal{U}} : \mathcal{U} \subseteq \mathcal{G}$ global bisection$\} \leq Homeo(X)$.

Examples Let $T \in \mathrm{Homeo}(X)$.

$$\mathbb{Z} \to \mathrm{Homeo}(X), \; n \mapsto T^n$$

$$\mathbb{Z} \ltimes_T X$$

When T is minimal (no non-trivial closed invariant subsets, orbits are dense)

$[[\mathbb{Z} \ltimes_T X]]$ has been extensively studied

- $[[\mathbb{Z} \ltimes_{T_1} X]] \cong [[\mathbb{Z} \ltimes_{T_2} X]]$ iff
  T_1, T_2 are flip conjugate
  $T_1 = S T_2 S^{-1}$ or $S T_2^{-1} S^{-1}$, $S \in \mathrm{Homeo}(X)$
  (Giordano-Putnam-Skau, 99)

- The derived subgroup $D([[\mathbb{Z} \ltimes_T X]])$ is finitely generated & simple.
  (Matui, 06)

- $[[\mathbb{Z} \ltimes_T X]]$ is amenable (Juschenko, Monod '12).

Let $G = (V, E, i, t)$ be a finite directed graph which is strongly connected, not a cycle.

For any $v, w \in V$, there is a directed path $v \longrightarrow w$.

Let

$$X = \{ x = (x_i)_{i \in \mathbb{N}} \in E^{\mathbb{N}} : t(x_k) = i(x_{k+1}) \}$$

with the relative product topology is a Cantor set.

Define $\sigma : X \to X$

$$\sigma(x)_i = x_{i+1}$$

Define

$$\mathcal{G}_G = \{(x, m-n, y) : x, y \in X, m, n \in \mathbb{N}, \sigma^m(x) = \sigma^n(y)\}.$$

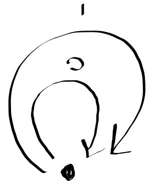
$$\subseteq X \times \mathbb{Z} \times X.$$

$$(x, m-n, y)(y, k-\ell, z) = (x, (m+k)-(n+\ell), z)$$

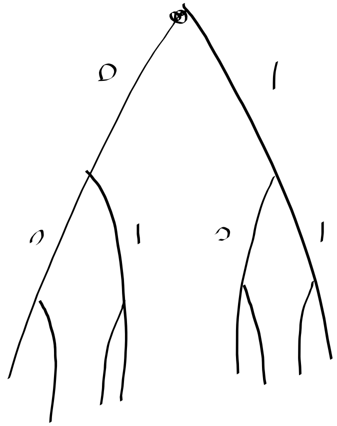
$$(x, m-n, y)^{-1} = (y, n-m, x), \quad r(x, m-n, y) = x$$

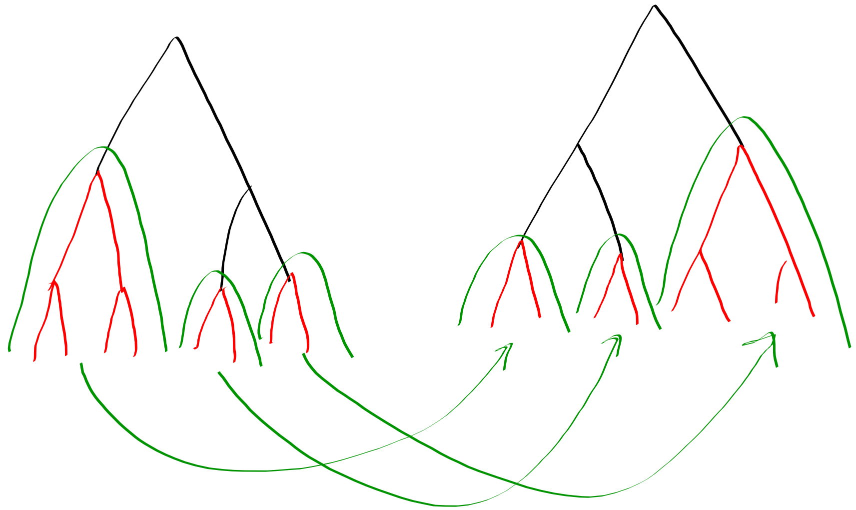
$$s(x, m-n, y) = y.$$

$$\mathcal{G}_G^{(0)} \cong X$$

Let G  $[[\mathcal{G}_a]] = V$ - Thompson's group

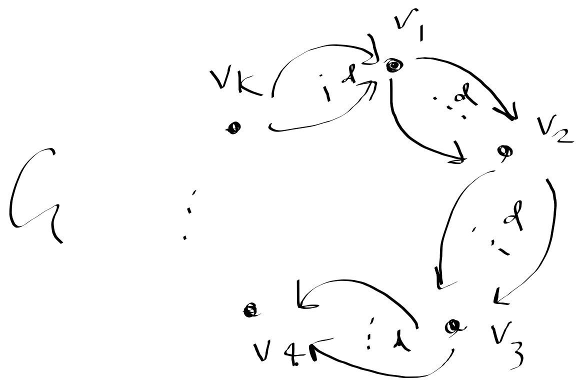
$X = \partial T_2$

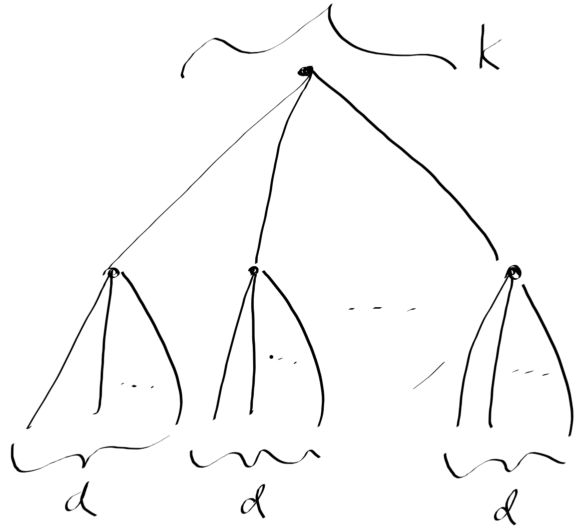


 $= \pi_a$.

$Z(\lambda) = \{\lambda x : \lambda$ finite path$\}$

$$U = Z(0) \times \{-1\} \times Z(00) \cup Z(10) \times \{2-2=0\} \times Z(01)$$
$$\cup\ Z(11) \times \{1\} \times Z(1) \subseteq \mathcal{G}_a$$





$$[[G_G]] = V_{d,k}.$$