Topological full groups pt II.

Fix an étale groupoid $\mathcal{G}$ with $\mathcal{G}^{(0)}$ a Cantor set

G discrete group, $c : \mathcal{G} \to G$ continuous groupoid hom.

Example $G \curvearrowright X$

$\mathcal{G} = X \rtimes G$

$c : (x, g) \longmapsto g.$

Given $g \in G$, say g is <u>global</u> if $s(c^{-1}(g)) = \mathcal{G}^{(0)} = r(c^{-1}(g))$ & <u>univalent</u> if $c^{-1}(g) \subseteq \mathcal{G}$ is a bisection.

$H \leq G$ is global / univalent if h is global / univalent for all $h \in H$.

$H \leq G$ is <u>faithful</u> if $\forall h \in H \setminus \{e\}$, there exists $\gamma \in c^{-1}(h)$ such that $s(\gamma) \neq r(\gamma)$.

If H is faithful, global & univalent, then H embeds in $[[\mathcal{G}]]$ by $h \longmapsto \pi_{c^{-1}(h)}$.

Commensurated subgroups

$H, K \leq G$ are commensurate if $|H : H \cap K|, |K : H \cap K| < \infty$.

$H \leq G$ is commensurated if H, gHg^{-1} commensurate for all $g \in G$.

Lemma $H \leq G$ is commensurated if & only if, for any $g \in G$, there exists $K \leq H$, such that $gKg^{-1} \leq H$ & K, gKg^{-1} finite index in H.

Proof ($\Rightarrow$) Let $K = H \cap g^{-1}Hg$.

($\Leftarrow$) $K \leq H \cap g^{-1}Hg \Rightarrow gKg^{-1} \leq gHg^{-1} \cap H$

$$|H : H \cap g^{-1}Hg| \leq |H : K| < \infty$$

$$|g^{-1}Hg : H \cap g^{-1}Hg| = |H : gHg^{-1} \cap H|$$

$$= |H : gKg^{-1}| < \infty.$$

Let $G \curvearrowright X$ Cantor set, $U \subseteq X$ clopen

$$\mathrm{rist}_G(U) = \{g \in G : g \text{ fixes } X \setminus U \text{ pointwise}\}$$
$$= \{g \in G : \mathrm{supp}\, g \subseteq U\}$$

If U, V are disjoint, $\mathrm{rist}_G(U)$ & $\mathrm{rist}_G(V)$ commute

G is called <u>virtually decomposable</u> if, for every clopen partition $\{U_1, \ldots, U_n\}$ of X

$$H = \langle \mathrm{rist}_G(U_i) \mid 1 \leq i \leq n \rangle$$

has finite index in G.

G virtually decomposable $\Rightarrow$ G is precompact in $\mathrm{Homeo}(X)$.

Prop $H \leqslant \mathcal{G}$ is faithful, global & univalent wrt $c : \mathcal{G} \to G$. If H is commensurated in G & $H \leqslant [[\mathcal{G}]]$ is virtually decomposable then H is commensurated in $[[\mathcal{G}]]$.

Proof Fix $g = \pi_U \in [[\mathcal{G}]]$.

Write $U = \dot{\bigcup}_{i=1}^{n} U_i$, $i \neq j \Rightarrow U_i \cap U_j = \emptyset$, $c(U_i) = \{g_i\}$.

$$g \cdot x = g_i x \quad \text{when} \quad x \in s(\mathcal{U}_i)$$

$$L = \prod \mathrm{rist}_H(s(\mathcal{U}_i)) \leq [[\mathcal{G}]]$$

& let $K = \bigcap_{i=1}^{n} g_i L g_i^{-1}$

Fix $k \in K$, $x \in \mathcal{G}^{(0)}$. Then $x \in s(\mathcal{U}_i)$

$$\begin{aligned} g^{-1} k g \cdot x &= g^{-1} k g_i \cdot x \\ &= g^{-1} (g_i k_i g_i^{-1}) g_i \cdot x \quad \text{where } k_i \in L \end{aligned}$$

$$g_i^{-1} \underbrace{g_i \underbrace{k_i \cdot x}_{s(\mathcal{U}_i)}}_{r(\mathcal{U}_i)}$$

$$= k_i \cdot x.$$

So $g^{-1} K g \leq L \leq H$.

We need to show K has finite index in H.

$|H : L| < \infty$ since H is virtually decomposable

$|L:K| < \infty$ because $H \leq G$ is commensurated,

$K = \bigcap g_i L g_i^{-1}$.

So $H \leq [[\mathcal{G}]]$ is commensurated.

$\langle \overline{H}, [[\mathcal{G}]] \rangle \leq \mathrm{Homeo}(\mathcal{G}^{(0)})$ is a tdlc group,

$[[\mathcal{G}]]$ is dense.