Free Products of Graphs

- Produce locally finite graphs non-discrete aut. gps.
- Includes known constructions
  - Cayley graphs of RAAG where graph has complete components.
  - trees with edge colourings.

Let $\Gamma_1, \dots, \Gamma_n$ be simple connected graphs

$$\Gamma_i = (V_i, E_i)$$

Choose $\tilde{u}_0 = (u_1, \dots, u_n) \in V_1 \times \dots \times V_n$

$\Gamma = (V, E)$ is the <u>free product</u> of $\Gamma_1, \dots, \Gamma_n$

$V = \bigcup_{\ell=0}^{\infty} V^{(\ell)}$ where $V^{(\ell)}$ defined recursively.

We will also $u : V \longrightarrow V_1 \times \dots \times V_n$ recursively.

$u(\tilde{v}) = (u_1(\tilde{v}), u_2(\tilde{v}), \dots, u_n(\tilde{v}))$

$V^{(\ell)}$ will be a set of strings of vertices in $V_1 \cup \dots \cup V_n$.

$V^{(0)} = \{\emptyset\}$ $u(\emptyset) = \tilde{u}_0$. Suppose that $V^{(\ell)}$ has been defined.

$$V^{(\ell+1)} = \{ v_1 \dots v_\ell v_{\ell+1} : v_1 \dots v_\ell \in V^{(\ell)};\ v_\ell \in V_i\ v_{\cdot} \in V_j \text{ with } i \neq j; \text{ and } v_{\ell+1} \neq u_j(v_1 \dots v_\ell) \}$$

Define $u_i(v_1 \dots v_\ell v_{\ell+1}) = \begin{cases} u_i(v_1 \dots v_\ell) & \text{if } i \neq j \\ v_{\ell+1} & \text{if } i = j \end{cases}$

$v_e \in V_i,\ v_{e+1} \in V_j$

$$E = \Big\{ \{\phi, v_1\} : \{u_j(\phi), v_1\} \in E_j \text{ for some } j \Big\}$$

$$\cup \Big\{ \{\tilde{v}, \tilde{v} v_{e+1}\} : \{u_j(\tilde{v}), v_{e+1}\} \in E_j,\quad i \neq j \Big\}$$

$\tilde{v} = v_1 \cdots v_e$

$$\cup \Big\{ \{v_1 \cdots v_{e-1} v_e,\ v_1 \cdots v_{e-1} v_e'\} : \{v_e, v_e'\} \in E_i \Big\} \quad (*)$$

<u>Note:</u> In $(*)$, if $v_e' = u_i(v_1 \cdots v_{e-1})$, then $v_1 \cdots v_{e-1} v_e' \notin V^{(e)}$.

We use $v_1 \cdots v_{e-1}$ instead.

The number of neighbours of $\tilde{v} = v_1 \dots v_e$ is

$$N_{\Gamma}(\tilde{v}) = N_{\Gamma_1}(u_1(\tilde{v})) + N_{\Gamma_2}(u_2(\tilde{v})) + \dots + N_{\Gamma_n}(u_n(\tilde{v}))$$

<u>Prop^n</u> Γ is connected.

<u>Prop^n</u> $u : \Gamma \longrightarrow \Gamma_1 \times \dots \times \Gamma_n$ is a surjective graph hm.

For $v_1 \in V^{(1)}$ define

$$\varphi(v_1) = \begin{cases} v' v_1, & \text{if } v_1 \in V_j,\ i \neq j \\ v_1' \cdots v_{e-1}' v_1, & \text{if } v_1 \in V_i \text{ and } v_1 \neq u_i(v_1' \cdots v_{e-1}') \\ v_1' \cdots v_{e-1}', & \text{if } v_1 = u_i(v_1' \cdots v_{e-1}') \end{cases}.$$

$\varphi(v_1) \in V'$ and $u(\varphi(v_1)) = u(v_1)$

Repeat.

$\underline{\varphi \text{ is onto}}$ Let $\tilde{w}' \in V'$, $\tilde{w}' = w_1' \cdots w_n'$.

<u>φ is one-to-one</u>

Construct $\bar{\varphi}'$ by same argument changing roles of Γ & Γ'.

<u>φ is a graph hm</u>

Check.

$\blacksquare$

<u>Prop^n</u> There is a hm $x \mapsto \hat{x} : \mathrm{Aut}(\Gamma_i) \to \mathrm{Aut}(\Gamma)$.

Propⁿ Let Γ, Γ' be defined as above with initial values $\tilde{u}_0, \tilde{u}_0'$ resp. Then $\Gamma \cong \Gamma'$.

Proof: Define $\varphi : \Gamma \to \Gamma'$ recursively.

Since $\Gamma_1, \dots, \Gamma_n$, there is $\tilde{v}' \in V(\Gamma')$ with $u(\tilde{v}') = \tilde{u}_0$.

Suppose that $\tilde{v}' = v_1' \dots v_\ell'$ where $v_\ell' \in V_i$

Put $\varphi(\emptyset) = \tilde{v}'$.

If $m \geq \ell$ and $\tilde{w}' = v_1' \cdots v_\ell' w_{\ell+1}' \cdots w_m'$,

then $\tilde{w}' = \varphi(w_{\ell+1}' \cdots w_m')$, where $w_{\ell+1}' \cdots w_m' \in V$.

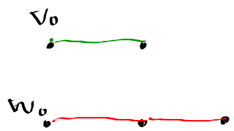
Suppose that $\tilde{w}' = v_1' \cdots v_{\ell-1}' w_\ell' \cdots w_m'$

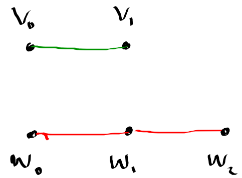
with $\underset{m=\ell-1}{w_\ell' \cdots w_m'} = \emptyset$, $w_\ell' \neq v_\ell'$.

In the first case $\varphi(u_i(v_1 \cdots v_{\ell-1}')) = w'$

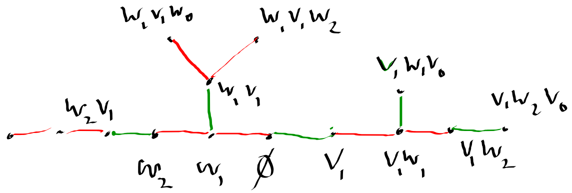
In the second case, if $w_\ell' \notin V_i$, then

$$w' = \varphi(u_i(v_1' \cdots v_{\ell-1}') w_\ell' \cdots w_m')$$



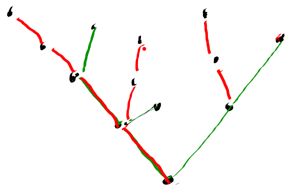


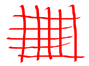
$\tilde{U}_0 = (V_0, W_0)$



$U(w_1) = (V_0, w_1)$ $U(V_1) = (V_1, W_0)$

$U(W_2) = (V_0, W_2)$



Γ_1  Γ_2