Embedding totally disconnected locally compact groups into simple groups

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The class $\mathscr{S}$ of compactly generated, topologically simple locally compact groups can be split into three parts:

- $\mathscr{S}_{\mathrm{Lie}}$ the simple Lie groups;
- $\mathscr{S}_{\mathrm{disc}}$ the finitely generated simple groups (with discrete topology);
- $\mathscr{S}_{\mathrm{td}}$ the compactly generated, topologically simple groups that are totally disconnected locally compact (t.d.l.c.), but not discrete.

Up to isomorphism, $\mathscr{S}_{\mathrm{Lie}}$ is countable and its members have been explicitly listed.
$\mathscr{S}_{\mathrm{disc}}$ has $2^{\aleph_0}$ isomorphism types and these are effectively unclassifiable (there is no hope to classify even the $\aleph_0$ finitely *presented* simple groups).
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Some sources of examples of groups in $\mathscr{S}_{\mathrm{td}}$:

- algebraic groups over $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$;
- completions of Kac–Moody groups over finite fields;
- groups specified by local actions on trees and buildings;
- commensurators of profinite branch groups.

(Smith 2017) There are $2^{\aleph_0}$ pairwise nonisomorphic (as abstract groups) groups in $\mathscr{S}_{\mathrm{td}}$.
Open question: Are there uncountably many *local* (= in a neighbourhood of the identity) isomorphism classes?

A compactly generated t.d.l.c. group G is **expansive** if there is $U \leq G$ open with $\bigcap_{g \in G} gUg^{-1} = \{1\}$. (Every $G \in \mathscr{S}_{\mathrm{td}}$ is expansive.) Equivalently, G acts faithfully continuously vertex-transitively on a connected locally finite graph. Are there uncountably many local isomorphism types of such groups?

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Theorem (Hall, Goryushkin, Schupp, 1974–76) Every countable group embeds in some 2-generator simple group.

Is every second-countable t.d.l.c. group G an open subgroup of some $L \in \mathscr{S}_{\mathrm{td}}$?

First "no": consider ($n \geq 2$)

$$G = \prod_{p \text{ prime}} (\mathrm{PSL}_n(\mathbb{Q}_p), \mathrm{PSL}_n(\mathbb{Z}_p)); \; U = \prod_{p \text{ prime}} \mathrm{PSL}_n(\mathbb{Z}_p).$$

G is expansive, but does not embed in any compactly generated t.d.l.c. group; U is compact, but does not embed in any compactly generated expansive t.d.l.c. group. (Proof idea: consider how U would act on a connected locally finite graph...) Some work has been done on embeddability into compactly generated (expansive) groups (e.g. by Caprace–Cornulier), but it is wide open in general.

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Second “no”: there are additional local restrictions on groups $L \in \mathscr{S}_{\mathrm{td}}$, e.g. L cannot have any nontrivial abelian subgroup with open normalizer (Caprace–R.–Willis).
To give us flexibility on the local structure, let's say we just want an open subgroup $K \rtimes G$ of L where K is compact.

Third “no”: $L \in \mathscr{S}_{\mathrm{td}}$ is unimodular, so every open subgroup of L is unimodular; but G need not be, and if G is not unimodular then neither is $K \rtimes G$ for G compact.
So let us go slightly beyond $\mathscr{S}_{\mathrm{td}}$ to a class that allows non-unimodular groups L (but still with a “large” normal subgroup in $\mathscr{S}_{\mathrm{td}}$).

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Theorem 1 (Garrido–R.)

Let G be a compactly generated expansive t.d.l.c. group. Then there is a t.d.l.c. group L and an open subgroup O of L with the following properties:

(i) $O \cong K \rtimes G$, where K is compact;

(ii) the derived group $\mathrm{D}(L)$ of L is open and belongs to $\mathscr{S}_{\mathrm{td}}$;

(iii) $L = \mathrm{Aut}(\mathrm{D}(L)) = \mathrm{D}(L)G\langle s\rangle$ where $s^2 = 1$.

Corollary

Given a finitely generated subgroup F of $\mathbb{Q}^*_{>0}$, there is $S \in \mathscr{S}_{\mathrm{td}}$ such that F is the image of the modular function of $\mathrm{Aut}(S)$.

The construction is to form a suitable group acting on a countable tree, and then take the piecewise full group of its action on a compactification of the boundary of the tree.

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A group H acting vertex-transitively on a tree T has a **local action** given by H_v acting on the edges incident with v. For a transitive permutation group P, there is a group $\mathbf{U}(P)$ acting vertex-transitively on a tree with local action P, such that every other such group is conjugate to a subgroup of $\mathbf{U}(P)$ (Burger–Mozes 2000, Smith 2017). This falls into the general framework of local actions on trees developed by R.–Smith.

First step towards Theorem 1: let G act on G/U for a compact open subgroup U such that $\bigcap_{g \in G} gUg^{-1} = \{1\}$, and form $H = \mathbf{U}(G)$. This has some of the properties we want:

- H has vertex stabilizer $H_v \cong K \rtimes G$, where K is the fixator of the 1-ball around v. K is compact and H_v is open.
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By a result of Tits, H also has an open simple (or trivial) subgroup H^+ generated by the arc stabilizers. But it is not quite the simple group we want: H^+ has local action G^+, where G^+ is generated by the point stabilizers of G, and $H/H^+ \cong G/G^+ * C_2$. If G is not generated by point stabilizers, then H/H^+ is nonabelian and H^+ is not compactly generated.

To get the right group in $\mathscr{S}_{\mathrm{td}}$, we appeal to some general results about piecewise full groups obtained by Garrido–R.–Robertson. Let G be a group acting by homeomorphisms on the Cantor set X. The **piecewise full group** $\mathrm{F}(G)$ is the group of homeomorphisms $h \in \mathrm{Homeo}(X)$ such that for all $x \in X$, there is $g_x \in G$ and a neighbourhood O_x of x such that $h|_{O_x} = g_x|_{O_x}$. To obtain the group L in Theorem 1, we appeal to general results about such groups.

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Definition

Let X be the Cantor set and let G be a topological group acting faithfully by homeomorphisms on X. The action of G is:

- **minimal** if every orbit is dense;
- **expansive** if the topology of X is generated by the G-translates of a finite set of clopen sets;
- **locally decomposable** if for every clopen partition $\mathcal{P}$ of X, the subgroup $\langle \mathrm{rist}_G(Y) \mid Y \in \mathcal{P} \rangle$ is open.

Theorem 2 (Garrido–R.–Robertson)

Let H be a t.d.l.c. group with a faithful minimal expansive locally decomposable action by homeomorphisms on the Cantor set, such that the rigid stabilizers are not discrete. Then the topology of H extends to the piecewise full group $\mathrm{F}(H)$, with H as an open subgroup. Moreover, $\mathrm{D}(\mathrm{F}(H))$ is open in $\mathrm{F}(H)$ and belongs to $\mathscr{S}_{\mathrm{td}}$, and we have $\mathrm{Aut}(\mathrm{D}(\mathrm{F}(H))) = \mathrm{F}(H)$.

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Next steps: find an action of $H = \mathbf{U}(G)$ on the Cantor set, and check it has the right dynamical properties.
H acts on a tree T. There is a natural topology on the space of ends ∂T induced by the following metric: start at a base point v_0, and say the rays $(v_0, v_1, \dots)$ and $(v_0, w_1, \dots)$ have distance 2^{-i} if they first differ in the i-th entry. However, if v_0 (or any other vertex) has ∞ neighbours, clearly this space is not compact.

Let AT be the set of arcs. Given $a \in AT$ then $T - a$ divides into two **half-trees**, where T_a has the vertices closer to $t(a)$ and $T_{\overline{a}}$ has the vertices closer to $o(a)$. We then call the set ∂T_a of ends of T_a a **half-space** Y_a of ∂T.

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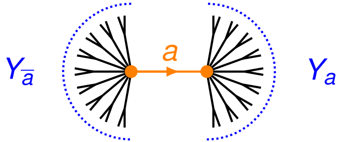

Let $\mathcal{H}$ be the set of half-spaces and define $\iota : \partial T \to \{0, 1\}^{\mathcal{H}}$ by setting $\iota(\omega)(Y_a) = 1$ if $\omega \in Y_a$ and 0 otherwise; we then define $\overline{\partial T} = \overline{\iota(\partial T)}$. In other words: a point in $\overline{\partial T}$ is an "ultrafilter of half-spaces of ∂T". This is in fact a topological embedding of ∂T into $\overline{\partial T}$ and the action of H extends to $\overline{\partial T}$ by continuity; moreover, $\overline{\partial T}$ is a Cantor set.

Half-spaces $\overline{Y}_a := \overline{\iota(Y_a)}$ of $\overline{\partial T}$ do not form a base of topology in general, only a subbase. However, every clopen partition of $\overline{\partial T}$ can be refined to a partition $\mathcal{P}_{T'}$ for a finite subtree T', where the parts of $\mathcal{P}_{T'}$ correspond to the preimages of the closest point projections of VT onto T'. Each part is then a finite intersection of half-spaces.

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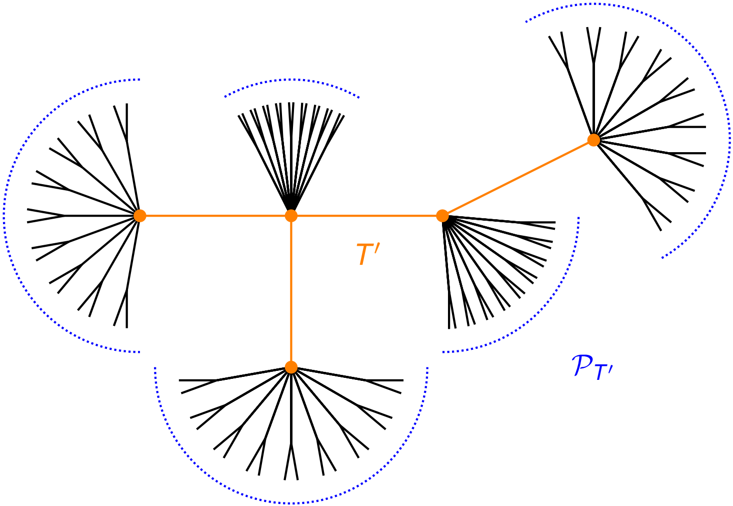
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- Faithful action: the tree has enough ends that $\mathrm{Aut}(T)$ acts faithfully on ∂T.
- Expansive action: the topology of $\overline{\partial T}$ is generated by half-spaces, and H acts transitively on these.
- Minimal action: every nonempty open subspace O contains a half-space, so the H-translates of O cover the space.
- Locally decomposable, no discrete rigid stabilizer: It is enough to consider clopen partitions $\mathcal{P}_{T'}$. Here one sees that the pointwise stabilizer of T' in H (which is open) is the product of the rigid stabilizers, due to how H is defined by local actions. Moreover, each rigid stabilizer contains a rigid stabilizer of a half-space; the latter are nondiscrete as long as G has nontrivial point stabilizers.

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We now take $L = \mathrm{F}(H)$, and this has the right properties for Theorem 1 (subject to a minor adjustment to get nontrivial point stabilizers when G is discrete). The last part of Theorem 1 was that the abelianization $L_{\mathrm{ab}} := L/\mathrm{D}(L)$ is accounted for by G plus an element of order 2.

Here we use the normal subgroups $\mathrm{S}(H)$ and $\mathrm{A}(H)$ of $\mathrm{F}(H)$ introduced by Nekrashevych, which are generated respectively by “transpositions” and “3-cycles” s on disjoint clopen parts of the Cantor set, where on each part s acts as some element of H. Nekrashevych showed under quite general circumstances that $\mathrm{A}(H)$ is simple (so in our case, $\mathrm{A}(H) = \mathrm{D}(L) \geq H^+$).

Using arguments specific to the present situation, we show $\mathrm{S}(H) = \mathrm{A}(H)\langle s \rangle$ for any edge inversion s and then $L = \mathrm{S}(H)H_v$; writing $H_v = K \rtimes G$, we have $L = \mathrm{S}(H)G$.

When the tree is locally finite, actually $L = \mathrm{S}(H)$; this is a special case of results of Lederle.
On a locally infinite tree, we can get a lower bound on $L/\mathrm{S}(H)$.
Given $h \in H$ write $[h]$ for its image in the abelianization H_{ab}. ($H_{\mathrm{ab}} = G/\mathrm{D}(G)G^+ \times C_2$.) Given $g \in L$, there is a finite subtree T' such that on each part Z_v of $\mathcal{P}_{T'}$ ($v \in VT'$), g acts as an element g_{Z_v} of H. There is then a homomorphism θ from L to H_{ab} given by

$$\theta(g) = \prod_{v \in VT'} [g_{Z_v}]^{(2-\deg_{T'}(v))}.$$

Given $g \in H$, then $\theta(g) = [g]^2$. So we have a short exact sequence

$$1 \to E \to H_{\mathrm{ab}} \to L_{\mathrm{ab}} \to 1$$

where E has exponent ≤ 2.

When the tree is locally finite, actually $L = \mathrm{S}(H)$; this is a special case of results of Lederle.
On a locally infinite tree, we can get a lower bound on $L/\mathrm{S}(H)$.
Given $h \in H$ write $[h]$ for its image in the abelianization H_{ab}. ($H_{\mathrm{ab}} = G/\mathrm{D}(G)G^+ \times C_2$.) Given $g \in L$, there is a finite subtree T' such that on each part Z_v of $\mathcal{P}_{T'}$ ($v \in VT'$), g acts as an element g_{Z_v} of H. There is then a homomorphism θ from L to H_{ab} given by

$$\theta(g) = \prod_{v \in VT'} [g_{Z_v}]^{(2-\deg_{T'}(v))}.$$

Given $g \in H$, then $\theta(g) = [g]^2$. So we have a short exact sequence

$$1 \to E \to H_{\mathrm{ab}} \to L_{\mathrm{ab}} \to 1$$

where E has exponent ≤ 2.

The following example shows how to deduce the corollary. Define the modular function as

$$\Delta_G(g) := \frac{\mu(gUg^{-1})}{\mu(U)},$$

where μ is a right-invariant Haar measure on G.

Example

Let $F = \langle g_1, \ldots, g_m \rangle \leq \mathbb{Q}^*_{>0}$ and let $p_1, \ldots, p_n$ be the primes involved in $g_1, \ldots, g_m$. Set $G = \prod_{i=1}^n \mathbb{Q}_{p_i} \rtimes F$, where g_i acts on $\mathbb{Q}_{p_j}$ as multiplication by g_i^{-1}; then $\Delta_G(G) = F$. Note that $U = \prod_{i=1}^n \mathbb{Z}_{p_i}$ has trivial core in G and $G = \langle U, F \rangle$.
Let G act on G/U and form $L = \mathrm{F}(\mathbf{U}(G))$ and $S = \mathrm{D}(L)$. Then $L = \mathrm{Aut}(S)$ with $S \in \mathscr{S}_{\mathrm{td}}$, and $L = SG\langle s \rangle$ where $s^2 = 1$, so $\Delta_L(L) = \Delta_L(G)$. Moreover, $\Delta_L(G) = \Delta_G(G) = F$, since G is contained in an open subgroup of the form $K \rtimes G$ where K is compact.