COMPACTLY PRESENTED GROUPS

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Abstract. This survey purports to be an elementary introduction to compactly presented groups, which are the analogue of finitely presented groups in the broader realm of locally compact groups. In particular, compact presentation is interpreted as a coarse simple connectedness condition on the Cayley graph, and in particular is a quasi-isometry invariant.

In the appendix, an example of a Lie group, not quasi-isometric to any homogeneous graph, is given; the short argument relies on results of Trofimov and Pansu, anterior to 1990.

1. Introduction

In geometric group theory, the principal object of study is often a discrete finitely generated group G. Its geometry is the geometry of its Cayley graph $\Gamma(G, S)$, where S is a finite generating set, whose vertices are elements of G and edges are of the form $\{g, gs\}$ for $g \in G, s \in S \cup S^{-1} - \{1\}$. It has long been known that the study of the geometry of those groups can be eased when G sits as a cocompact lattice in a non-discrete locally compact group, like a a connected Lie group. This holds, for instance, when G is a finitely generated torsion-free nilpotent group, or more generally a polycyclic group (see [Rag, Chapters 2–4]). For some classes of solvable groups, for instance a matrix group over $\mathbf{Z}[1/n]$, the target group has to be a linear group over a product of local fields. It thus appears that the natural setting for the geometric study of groups is to consider compactly generated locally compact groups, the compact generation reducing to finite generation in the case of discrete groups. In the locally compact setting, the Cayley graph with respect to a compact generating subset appears at first sight as awful as for instance it has infinite degree; however its large scale geometry has a reasonable behaviour: for instance, it is always quasi-isometric to a graph of bounded degree (obtained by restricting to a "separated net", see [Gr93, §1.A]), although this graph cannot always be chosen homogeneous (see Appendix A).

Compactly presented groups generalize to locally compact groups what finitely presented groups are to discrete groups. On the other hand they are far less well-known[1], although they were introduced and studied by the German School in the sixties and seventies [Kn, Behr, Ab].

A locally compact group G is *compactly presented* if it has a compact subset S such that G, as an abstract group, has a presentation with S as a set of generators, and relators of bounded length. For instance, a discrete group is compactly presented if and only if it is finitely presented.

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[1]A Google search on February 2, 2009 gives 13500 occurences for "finitely presented group" and only 5 for "compactly presented group", including quotation marks in both cases.

Other compactly presented locally compact groups include connected Lie groups, reductive algebraic groups over local fields, but not all algebraic groups over local field: for instance, if $\mathbf{K}$ is any non-Archimedean local field, then the compactly generated group $\mathrm{SL}_2(\mathbf{K}) \ltimes \mathbf{K}^2$ is not compactly presented (see Example 5.5), although compactly generated. An extensive and thorough study led Abels [Ab2] to characterize linear algebraic groups over a local field of characteristic zero that are compactly presented. In contrast, the case of finite characteristic is not settled yet.

The following note intends to describe the basics on compactly presented groups, with the hope to make them better known. Properties (1), (2), (4) below were obtained in [Ab]; we obtain here these results with in mind the geometric point of view initiated by Gromov.

(1) Being compactly presented is preserved by extensions, and by quotients by closed normal subgroups that are compactly generated as normal subgroups (Lemmas 3.5 and 2.8).
(2) Any compactly generated locally compact group has a covering which is a compactly presented locally compact group (a covering of G means a locally compact group H whose quotient by some normal discrete subgroup is isomorphic to G) (Proposition 3.7).
(3) Among compactly generated locally compact groups, to be compactly presented is a quasi-isometry invariant (Corollary 4.11). In particular, it is inherited by and from closed cocompact subgroups.
(4) If, in an exact sequence of locally compact groups $1 \to N \to G \to Q \to 1$, the group Q is compactly presented and G is compactly generated, then N is compactly generated as a normal subgroup of G. In particular, if N is central, then it is compactly generated. (Proposition 4.15)

Section 2 introduces the new and more general notion of groups *boundedly generated* by some subset. Compactly presented groups are introduced in Section 3. In Section 4, we explain the link between simple connectedness and compact presentedness, which notably allows to obtain the two latter results (3) and (4). Some further examples are developed in Section 5.

2. Boundedly presenting subsets

Definition 2.1. Let G be a group and $S \subset G$ a subset. We say that G is *boundedly presented* by S if G has a presentation with S as set of generators, and relations of bounded length.

Example 2.2.

- If S is finite, then G is finitely presented if and only if G is boundedly presented by S.
- The group G is always boundedly presented by itself. Indeed, as set of relations one can choose relations of the form $gh = k$, $g, h, k \in G$, which have length 3.

Definition 2.3. Let G be a group and S a generating subset. We say that S is a *defining* (generating) subset if G is presented with S as set of generators, subject to the relations $gh = k$ for $g, h, k \in S$.

Clearly, if S is a defining generating subset of G, then G is boundedly presented by S. This has a weak converse [Behr, Hilfssatz 1]. Write $S^n = \{s_1 \dots s_n : s_1, \dots, s_n \in S\}$.

Lemma 2.4. *Let G be boundedly presented by a subset S. Then, for some n, S^n is a defining generating subset of G.*

Proof. Let G be presented by S as set of generators, with relations of length bounded by n. We claim that S^n is a defining generating subset of G.

Now let us work in the group $\tilde{G}$ presented with S^n as set of generators and subject to the relations of the form $gh = k$ for $g, h, k \in S^n$ whenever this equality holds in G. Let $u_1, \dots, u_m$ be elements in S^n, such that the relation $u_1 \dots u_m = 1$ holds in G. Write $u_i = v_{i1} \dots v_{in}$ in G, with all $v_{ij} \in S$. Then it is clear from the defining relations of $\tilde{G}$ that $u_i = v_{i1} \dots v_{in}$ holds in $\tilde{G}$, so that $u_1 \dots u_m = v_{11} \dots v_{1n} \dots v_{m1} \dots v_{mn}$ holds in $\tilde{G}$. Since $v_{11} \dots v_{1n} \dots v_{m1} \dots v_{mn} = 1$ holds in G and by the assumption on S, this element can be written (in the free group generated by S, hence in $\tilde{G}$) as a product of conjugates of elements w_k of length $\leq n$ with respect to S, such that $w_k = 1$ in G. Again it follows from the defining relations of $\tilde{G}$ that the equalities $w_k = 1$ hold in $\tilde{G}$. Accordingly $u_1 \dots u_m = 1$ holds in $\tilde{G}$. This shows that the natural morphism of $\tilde{G}$ onto G is bijective, so that the lemma is proved. □

Lemma 2.5. *Let G be a group and S a generating subset. Then G is boundedly presented by S if and only if G is boundedly presented by $S \cup S^{-1}$.*

Proof. Immediate from the definition. □

Lemma 2.6. *Let G be a group, and S_1, S_2 be generating subsets.*

(1) If G is boundedly presented by S_1^n for some $n \geq 1$, then it is boundedly presented by S_1

(2) If $S_1 \subset S_2 \subset S_1^n$ for some $n \geq 1$, and if G is boundedly presented by S_1, then it is boundedly presented by S_2.

(3) If $S_1^m \subset S_2^n \subset S_1^{m'}$ for some $m, m', n \geq 1$, then G is boundedly presented by S_1 if and only if it is boundedly presented by S_2.

Proof. (1) is obvious: take relators with bounded length with respect to generators in S_1^n, and, expressing elements of S_1^n as products of n elements of S_1, we get defining relators of bounded length with respect to generators in S_1.

(2) Consider a family (r_i) of relations with bounded length with respect to generators in S_1, defining a presentation of G. Consider the group $\tilde{G}$ presented with S_2 as set of generators, and subject to the relations on the one hand (r_i) and on the other hand of the form $u_1 \dots u_n = u$ for $x_1, \dots, x_n \in S_1$ and $u \in S_2$. We claim that the natural morphism of $\tilde{G}$ onto G is injective.

Consider an element $u_1 \dots u_k$ of $\tilde{G}$, with $u_i \in S_2$, and suppose that $u_1 \dots u_k = 1$ holds in G. Write u_i as a word of length $\leq n$ in elements of S_1. Then the fact that $u_1 \dots u_k = 1$ in $\tilde{G}$ follows from the relations r_i.

(3) Suppose G boundedly presented by S_1. Since $S_1 \subset S_1^m \subset S_1^m$, by (2) G is boundedly presented by S_1^m. Since $S_1^m \subset S_2^n \subset S_1^{mm'}$, by (2) again, G is boundedly presented by S_2^n. By (1), G is boundedly presented by S_2. Note that the hypothesis are symmetric in S_1 and S_2, so that we do not need to prove the converse. □

Remark 2.7. It is not true that if G is boundedly presented by S_1 and $S_2 \supset S_1$, then G is necessarily boundedly presented by S_2. Indeed, let G be equal to the infinite cyclic group $\mathbf{Z}$, set $u_n = n!$, and let $S = \{u_n : n \geq 1\}$. Then S generates G, which is finitely presented. However, G is not boundedly presented by S.

Suppose the contrary, i.e. that there is a family of relations between u_k's of length $\leq n-1$, with $n \geq 5$. We can add to these relations the relations $[u_k, u_\ell]$, and write all other ones as words of minimal length in the free *abelian* group generated by all u_k's. Now consider a relation r involving u_n, which is not a commutator. Permuting the letters in r if necessary, we can write $r = r_1 r_2 r_3$, where r_1 involves u_k's for $k < n$, r_2 involves u_n's, and r_3 involves u_k's for $k > n$. Then, in $\mathbf{Z}$, $|r_1| < n!$ while $r_2 r_3$ is divisible by $n!$. It follows that $r_1 = r_2 r_3$ in $\mathbf{Z}$. Now $r_2 = r_3^{-1}$ in $\mathbf{Z}$, but $|r_2| < (n+1)!$ and r_3 is divisible by $(n+1)!$ in $\mathbf{Z}$. This implies $r_2 = 1$. Since r has minimal length, this implies that u_n does not appear in r, a contradiction.

Lemma 2.8. *Let G be boundedly presented by a symmetric generating subset S. Let N be a normal subgroup. Suppose that N is generated, as a normal subgroup, by $N \cap S^n$ for some n. Then G/N is boundedly presented by the image of S.*

Lemma 2.9. *Let G be a group, and N a normal subgroup; denote $p : G \to G/N$ the natural projection. Suppose that N is boundedly presented by a subset T. Let $T' \subset G$ be such that $p(T') = T$. Then G is boundedly presented by $S = N \cup T'$.*

Proof. Let (r_i) be defining relations of bounded length for $(G/N, T)$. Let $\tilde{G}$ be defined by $T \cup S$ as set of generators, and subject to the following relations:

(i) all possible relations obtained by lifting relations $r_i(t_1, \ldots, t_n) = 1$, giving relations of the form $r_i(\tilde{t}_1, \ldots, \tilde{t}_n) = u$ (with $u \in N$),
(ii) relations of the form $tut^{-1} = v$ for $t \in T' \cup T'^{-1}$ and $u, v \in N$,
(iii) relations of bounded length defining (N, T).

Let us show that $p : \tilde{G} \to G$ is bijective. Note that it follows from the relations that N can considered as a normal subgroup of $\tilde{G}$. Let $u_1 \ldots u_n$ belong to $\tilde{G}$, such that $u_1 \ldots u_n = 1$ in G. Then $u_1 \ldots u_n = 1$ in G/N, so that we can write $u_1 \ldots u_n = \prod (g_k m_k g_k^{-1})$, where m_k's are elements among r_i's. Write $m_k = \rho_k v_k$, where ρ_k is a relation of G from (i) and $v_k \in N$. Then $\prod (g_k m_k g_k^{-1}) = \prod (g_k \rho_k g_k^{-1}) v$ for some $v \in N$ (this holds in the group only subject to the relations (ii)). Thus in $\tilde{G}$, $u_1 \ldots u_n = v$. Hence v is in the kernel of p, but p is injective in restriction to N. Accordingly $v = 1$ and $u_1 \ldots u_n = 1$ in $\tilde{G}$. ☐

Lemma 2.10. *Consider an extension $1 \to N \to G \to Q \to 1$. Let T resp. N boundedly present W resp. Q. Let W' be a subset of G whose projection into Q contains W. Suppose in addition that G has a Hausdorff topological group structure such that both T and W' are compact and T has non-empty interior in N. Then G is boundedly presented by $S = T \cup W'$.*

Proof. Let (r_i) be defining relations of bounded length for (Q, W). Let $\tilde{G}$ be defined by $T \cup S$ as set of generators, and subject to the following relations:

(i) relations obtained by lifting relations $r_i(t_1, \ldots, t_n) = 1$, giving relations of the form $r_i(\tilde{t}_1, \ldots, \tilde{t}_n) = u$ (with $u \in N$ expressed as a word in letters in $T \cup T^{-1}$),
(ii) relations of the form $wuw^{-1} = v$ for $w \in W' \cup W'^{-1}$, $u \in T$, $v \in N$.

Observe that the relations in (i) are of bounded length: if m is a bound to the length of r_i's, then the elements u's appearing in (i) belong to the intersection of N with the m-ball of G with respect to W', which is compact, hence is contained in the

m'-ball of N with respect to T for some m'. Similarly, the v appearing in (ii) has bounded length, so that relators in (ii) also have bounded length.

Let us show that $p : \tilde{G} \to G$ is bijective. As in the preceding proof, it follows from the relations that the subgroup $\tilde{N}$ of $\tilde{G}$ generated by T is normal in $\tilde{G}$. Let $u_1 \dots u_n$ belong to $\tilde{G}$, such that $u_1 \dots u_n = 1$ in G. As in the preceding proof, there exists $v \in \tilde{N}$ such that $u_1 \dots u_n = v$ in $\tilde{G}$. Now it follows from the relations (iii) that p is injective in restriction to $\tilde{N}$. Accordingly $v = 1$ and $u_1 \dots u_n = 1$ in $\tilde{G}$. □

3. Compactly presented groups

Definition 3.1. Let G be a topological group. We say that G is compactly presented if G is Hausdorff and there is a compact subset $S \subset G$ such that the (abstract) group G is boundedly presented by S.

The following is a standard application of compactness and of the Baire Theorem.

Lemma 3.2. *Let G be a locally compact group, and S a compact generating subset. Let $K \subset G$ be compact. Then $K \subset S^n$ for large n.* □

This has the following immediate consequence.

Lemma 3.3. *If a locally compact group G is compactly presented, then it is boundedly presented by all its compact generating subsets.* □

Remark 3.4. This is not true without the assumption that G is locally compact. Indeed, endow $\mathbf{Z}$ with the p-adic topology $\mathcal{T}_p$. Set $u_n = n!$ and $u_\infty = 0$. Then $\{u_n : n \leq \infty\}$ is a compact subset of $(\mathbf{Z}, \mathcal{T}_p)$. But we have shown in Remark 2.7 that $\mathbf{Z}$ is boundedly presented by $\{u_n : n \leq \infty\}$, while it is boundedly presented by $\{1\}$.

Proposition 3.5. *Let G be a locally compact group which is an extension of compactly presented groups. Then G is compactly presented.*

Proof. This follows from Lemma 2.10 using the fact that, if G is a locally compact group and N a closed normal subgroup, and if $p : G \to G/N$ denotes the projection, then for every compact subset W of G/N there exists a compact subset W' of G such that $p(W') = W$ (this is not true for general topological groups). □

Lemma 3.6. *Let G be a topological group. Let U be an open, symmetric generating subset containing* 1. *Consider the group $\tilde{G}$ presented with $\tilde{U} = U$ as set of generators, and with relations $gh = k$ for $g, h, k \in U$ whenever this relation holds in G. Then $\tilde{G}$ has a unique topology such that the natural projection $p : \tilde{G} \to G$ is continuous and $\tilde{U}$ is a neighbourhood of* 1 *in $\tilde{G}$ such that p induces a homeomorphism $\tilde{U} \to U$. Moreover, p is open with discrete kernel, and $\tilde{G}$ is Hausdorff whenever G is.*

Proof. Recall that a group topology is characterized by nets converging to 1. It is easily seen that for a topology on $\tilde{G}$ satisfying the conditions of the lemma, a net (x_i) converges to 1 if and only if eventually $x_i \in \tilde{U}$ and $p(x_i) \to 1$. This proves the uniqueness.

Now let us construct this topology. We define $\Omega \subset \tilde{G}$ to be open if, for every $x \in \Omega$, the set $p(x^{-1}\Omega \cap \tilde{U})$ contains a neighbourhood of 1 in G. This defines a topology: trivially, $\emptyset$ and $\tilde{G}$ are open and the class of open sets is stable under unions. Let Ω_1 and Ω_2 be open. Then, for every $x \in \Omega_1 \cap \Omega_2$, $p(x^{-1}\Omega_1 \cap \Omega_1 \cap \tilde{U}) =$

$p(x^{-1}\Omega_1 \cap \tilde{U}) \cap p(x^{-1}\Omega_2 \cap \tilde{U})$ (due to the injectivity of p on $\tilde{U}$), so that $\Omega_1 \cap \Omega_2$ is open.

Observe that, for such a topology, $\tilde{U}$ is open, and $x_i \to x$ if and only if $x^{-1}x_i \to 1$, while $x_i \to 1$ if and only if it satisfied the condition introduced at the beginning of the proof. In particular, $x_i \to 1$ if and only if $x_i^{-1} \to 1$, and, if $x_i \to 1$ and $y_i \to 1$, then $x_i y_i \to 1$. It remains to check that the topology is a group topology.

We first claim that, for every net (x_i) in $\tilde{G}$, if $x_i \to 1$ and $g \in \tilde{G}$, then $y_i = g x_i g^{-1} \to 1$. This is proved by induction on the length of g as a product of elements of $\tilde{U}$. This immediately reduces to the case when $g \in \tilde{U}$. In this case, for large i, $p(gx_i)$ and $p(gx_ig^{-1}) = p(y_i)$ belong to U. Moreover, $gx_i \to g$, hence eventually belongs to $\tilde{U}$. Since the relation $(gx_i)g^{-1} = \tilde{y}_i$ (where $\tilde{y}_i$ denotes the preimage of $p(y_i)$ in $\tilde{U}$) holds in G and $gx_i \in \tilde{U}$ for large i, by the definition of $\tilde{G}$, we obtain that $(gx_i)g^{-1} = \tilde{y}_i$ in $\tilde{G}$, so that $y_i = \tilde{y}_i \in \tilde{U}$ for large i. Now $p(y_i) \to 1$, and thus $y_i \to 1$.

Now let us prove that if $x_i \to x$, then $x_i^{-1} \to x^{-1}$. Indeed, $x^{-1}x_i \to 1$, and by conjugating by x, we obtain $x_i x^{-1} \to 1$. As observed above, we can take the inverse, so that $xx_i^{-1} \to 1$, i.e. $x_i^{-1} \to x^{-1}$.

Now let us prove that if $x_i \to x$ and $y_i \to y$, then $x_i y_i \to xy$. Indeed, $(xy)^{-1}x_i y_i = y^{-1}x^{-1}x_i y_i = y^{-1}(x^{-1}x_i y_i y^{-1})y$. By the combining the observations above, $y_i y^{-1} \to 1$, hence $x^{-1}x_i y_i y^{-1} \to 1$, hence $y^{-1}(x^{-1}x_i y_i y^{-1})y \to 1$. So this is a group topology.

It is immediate that p is continuous. Since $\tilde{U} \cap \mathrm{Ker}(p) = \{1\}$, the kernel is discrete. Moreover, it is immediate from the definition that p induces a homeomorphism of $\tilde{U}$ onto U. Since these are neighbourhoods of 1 in $\tilde{G}$ and G, this implies that p is open. If G is Hausdorff, then it is immediate from the definition that $\tilde{G} - \{1\}$ is open, so that $\tilde{G}$ is Hausdorff. □

As a particular case, we get

Proposition 3.7. *Let G be a compactly generated Hausdorff group. Then there exists a compactly presented group $\tilde{G}$ with a normal discrete subgroup N such that $G \simeq \tilde{G}/N$. Moreover, if G is locally compact, then so is $\tilde{G}$.*

Proposition 3.8. *Let G be a Hausdorff topological group, and K a compact normal subgroup. Then G is compactly presented if and only if G/K is.*

Proof. This immediately follows from Lemmas 2.9 and 2.8. □

The following lemma is due to Macbeath and Swierczkowski [MaS].

Lemma 3.9. *Let G be a topological Hausdorff group, and let H be a closed subgroup. Suppose that H is cocompact in G, i.e. there exists a compact subset $K \subset G$ such that $G = HK = \{hk : h \in H, k \in K\}$. Then G is compactly generated if and only if H is.*

Proof. Clearly, if H is generated by a compact subset S, then G is generated by $S \cup K$. Conversely, suppose G generated by a compact symmetric subset S; suppose also that $1 \in K$. Set $T = KSK^{-1} \cap H$ where $KSK^{-1} = \{k_1 s k_2^{-1} : k_1, k_2 \in K, s \in S\}$. Then T is compact, we claim that it generates H.

Let g belong to G, and write $g = s_1 \dots s_n$ with all $s_i \in S$. Set $k_0 = 1$, and define by induction $k_i \in K$, for $1 \le i \le n-1$ by: k_i is chosen in K so that $k_{i-1}s_i k_i^{-1} \in H$ (it exists because $HK = G$). Define $k_n = 1$. Then $g = \prod_{i=1}^{n} k_{i-1} g_i k_i^{-1}$. Since $g = \prod_{i=1}^{n-1} k_{i-1} g_i k_i^{-1} \in H$, $k_{n-1} g_n k_n^{-1} \in H$ if (and only if) $g \in H$. We thus obtain that H is generated by T. □

Remark 3.10. The proof also implies that, for $g \in H$, the length of g with respect to T is bounded by the length of g with respect to S. Hence, in H, the word length with respect to T and the length induced by the word length of G are equivalent, i.e. the embedding of H into G (endowed with these compact generating subsets) is a quasi-isometry.

Proposition 3.11. *Let G be a locally compact group, and let H be a closed cocompact subgroup. Suppose that there exists a compact subset $K \subset G$ such that $G = HK = \{hk : h \in H, k \in K\}$. Then G is compactly presented if and only if H is.*

The reader can try to prove this directly, but the proof is tedious and technical, especially the direction $\Rightarrow$. This will follow from Lemma 3.9, the subsequent remark, as well of results from the next section (Corollary 4.11).

4. Simple connectedness

4.1. **The Rips complex.** We denote by X_{top} the topological realization of a simplicial complex X.

Lemma 4.1. *Let X be a simplicial complex, with a vertex as base-point x_0. Let γ be a loop in X_{top} based on x_0. Then γ is homotopic to a combinatorial loop, i.e. a loop which is a sequence of consecutive edges travelled with constant speed.*

Proof. The reader can check it as an exercise, or refer to [Spa, Chap.3, Sec.6, Lemma 12]. □

Let Γ be a graph, and x_0 a base-point. Consider a combinatorial loop based on x_0, represented as a sequence $(x_0, x_1, \dots, x_n)$ where $x_n = x_0$ and $x_{i-1}x_i$ is an edge for $i = 1, \dots, n$. On the set of such loops, consider the equivalence relation generated by

$$(x_0, x_1, \dots, x_n) \sim (x_0, \dots, x_i, u, x_i, x_{i+1}, \dots, x_n)$$

whenever $\{u, x_i\}$ is an edge, which we call "graph-homotopic". Besides, given two paths $y = (y_1, \dots, y_n)$ and $z = (z_1 \dots, z_m)$, the composition yz is defined if $y_n = z_1$ and denotes the path $(y_1, \dots, y_n, z_2 \dots, z_m)$; similarly y^{-1} denotes $(y_n, \dots, y_1)$.

Let X be a simplicial complex. Consider paths joining given points x, x'. On this set, consider the equivalence relation generated by graph homotopies and

$$(x = x_0, \dots, x_n = x') \sim (x = x_0, \dots, x_i, u, x_{i+1}, \dots, x_n = x')$$

whenever $\{x_i, u, x_{i+1}\}$ is a 2-simplex; two paths equivalent for this equivalence relation are called combinatorially homotopic.

Lemma 4.2. *Let X be a simplicial complex. Let two combinatorial paths be homotopic (in the topological sense). Then they are combinatorially homotopic.*

Proof. This is [Spa, Chap.3, Sec.6, Theorem 16]. □

Lemma 4.3. *Let X be a simplicial complex, X^1 its 1-skeleton, and x_0 a base-point which is a vertex. Let $x = (x_0, x_1, \dots, x_n = x_0)$ be a loop. Suppose that x is, as a based loop in X_{top}, homotopically trivial. Then x is graph-homotopic to a loop of the form $\prod_{i=1}^m y_i r_i y_i^{-1}$, where y_i is a path from x_0 to a point z_i, and r_i is a loop based on z_i, such that all vertices of r_i belong to a common simplex in X, and such that the length of r_i is 3, i.e. r_i is of the form $(z_i, z'i, z''_i, z_i)$.*

Proof. This is a reformulation of Lemma 4.2. Indeed, suppose that $\{x_{i-1}, x_i, x_{i+1}\}$ is a 2-simplex. Then $(x_0, \dots, x_n = x_0)$ is graph-homotopic to

$$u(x_{i-1}, x_i, x_{i+1})u^{-1}(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n = x_0),$$

where $u = (x_0, \dots, x_{i-1}, x_{i+1})$. It then suffices to iterate this process. $\square$

Let X be a metric space. In all what follows, we consider X with the discrete topology (we study metric at large scale). Define the Rips complex of X as follows: for $t \in \mathbf{R}_+$, $R_t(X)$ is the simplicial complex with X as set of vertices, and $(x_1, \dots, x_d)$ is a simplex if $d(x_i, x_j) \leq t$ for all $i = 1, \dots, d$. It was used in particular by Gromov's monograph on hyperbolic groups [Gr87, Section 1.7].

Let G be a group and S a symmetric generating subset. Viewing G as a metric space (for the word length), we can define its Rips complex.

Lemma 4.4. *The embedding of G into $R_t(G)$ is a quasi-isometry.*

Proposition 4.5. *Let G be a group endowed with a generating subset S. The following conditions are equivalent.*

- *The group G is boundedly presented by S.*
- *$R_t(G)_{\rm top}$ is simply connected for sufficiently large t.*
- *$R_t(G)_{\rm top}$ is simply connected for some t.*

Proof. Suppose that $R_m(G)_{\rm top}$ is simply connected. Since the distance is integer-valued, we can suppose that m is integer (and $m \geq 1$: if $m = 0$ the only possibility is $G = \{1\}$). We endow G with the generating subset S^m, so that its Cayley graph coincides with the 1-skeleton of $R_m(G)_{\rm top}$. Replacing S by S^m if necessary, we assume that $m = 1$.

Let $s_1, \dots, s_m$ belong to S, such that $s_1 \dots s_n = 1$. Consider the combinatorial path $(1, s_1, s_1s_2, \dots, s_1 \dots s_{n-1}, 1)$: the corresponding path in $R_1(G)_{\rm top}$ is homotopically trivial. It follows from Lemma 4.3 that, in the free group generated by S (only subject to the relations of the form $ss^{-1} = 1$), $s_1 \dots s_n = \prod_{i=1}^k g_i r_i g_i^{-1}$ for some elements g_i, and some r_i such that, if we write $r_i = m_{i1}m_{i2}m_{i3}$ with $m_{ij} \in S$, then $r_i = 1$ in G. Therefore, we obtain that S is a defining subset of G.

Conversely, suppose that G is boundedly presented by S, by relations of length $\leq m$. Let us show that $R_{\lfloor m/2 \rfloor}(G)$ is simply connected. Consider a path γ in $R_{\lfloor m/2 \rfloor}(G)_{\rm top}$, based at 1, and let us show that it is homotopically trivial. By Lemma 4.1, we can suppose that γ is a combinatorial path

$$(1, s_1, s_1s_2, \dots, s_1s_2 \dots s_{n-1}, s_1s_2 \dots s_{n-1}s_n = 1).$$

By the presentation of G, we can write, in the free group generated by S (only subject to the relations of the form $ss^{-1} = 1$), $s_1 \dots s_n = \prod_i g_i r_i g_i^{-1}$, where r_i has length $\leq m$. To each element $u = u_1 \dots u_k$ of the semigroup freely generated by S, corresponds a combinatorial path $[u] = (1, u_1, u_1u_2, \dots u_1 \dots u_k)$. Then $[s_1 \dots s_n]$ and $[\prod_i g_i r_i g_i^{-1}]$ are graph-homotopic: this is a trivial consequence of the solution of the word problem in a free group. Now observe that, clearly, if two combinatorial based loops are graph-homotopic, then they are homotopic. Thus we are reduced to prove that $[\prod_i g_i r_i g_i^{-1}]$ is homotopically trivial. Clearly, this reduced to showing that each $[g_i r_i g_i^{-1}]$ is homotopically trivial. But this is freely homotopic to $[r_i]$, and this combinatorial loop, being of length $\leq m$, has all its vertices at distance $\leq \lfloor m/2 \rfloor$.

Hence it lies in the boundary of a simplex of $R_{\lfloor m/2 \rfloor}(G)$, and thus is homotopically trivial. $\square$

Let X be a metric space. A r-path in X is a sequence $(x_0, x_1, \dots, x_n)$ such that $d(x_i, x_{i+1}) \leq r$ for all i; it is said to join x_0 and x_n. If $x_0 = x_n$, it is called a r-loop based on x_0.

We introduce the r-equivalence between r-paths: this is the equivalence relation generated by $(x_0, \dots, x_n) \sim (x_0, \dots, x_i, y, x_{i+1}, x_n)$.

Definition 4.6. The metric space X is called coarsely connected if, for some r, every two points in X are joined by a r-path.

The metric space X is called coarsely simply connected if it is coarsely connected, and, for every r, there exists $r' \geq r$ such that every r-loop is r'-homotopic to the trivial loop.

Remark 4.7. Consider a geodesic circle C_R of length R. Let a r-path γ go round C_R. Then the reader can check that γ is r-homotopic to the trivial path if and only if $R \leq 3r$. In particular, a bunch of circles of radius tending to ∞ if not coarsely simply connected. Removing one point (other than the base-point) in each of these circles, we obtain a simply connected (and even contractible) metric space which is not coarsely simply connected.

Proposition 4.8. *Let X be a metric space. The following conditions are equivalent:*

(1) X is coarsely simply connected,

(2) For some t_0, $R_{t_0}(X)$ is connected, and for every t, there exists $t' \geq t$ such that every loop in $R_t(X)$ is homotopically trivial in $R_{t'}(X)$.

(3) For some r, $R_r(X)$ is simply connected.

Proof. (1)⇒(2) Suppose that X is coarsely simply connected. Clearly, $R_{t_0}(X)$ is connected for some t_0. Consider a loop in $R_r(X)$. By Lemma 4.1, it is homotopic to a combinatorial loop, defining a r-path on X. For some $r' \geq r$, this loop is r'-equivalent to the trivial loop. Thus the loop is homotopically trivial in $R_{r'}(X)$.

(2)⇒(1) Suppose that (2) is satisfied. Then X is t_0-connected. Fix $t \geq t_0$, and take t' as in (2). Consider a t-loop in X. Then the corresponding loop in $R_{t'}(X)$ is homotopically trivial. By Lemma 4.2, we obtain that it is t'-equivalent to the trivial loop.

(3)⇒(2). Consider a t-loop in X. Set $r' = \max(r, t)$. Since X is r-connected, this loop is r'-equivalent to a r-loop, defining a combinatorial loop on $R_r(X)$. Since $R_r(X)$ is simply connected and using Lemma 4.2, this r-loop is r-equivalent to the trivial loop. Thus our original loop is r'-equivalent to the trivial loop.

(2)⇒(3). Fix $t \geq t_0$, and t' as in (2). First note that $R_{t'}(X)$ is pathwise connected. Consider a loop in $R_{t'}(X)$. It is homotopic to a combinatorial loop. Since X is t-connected, it is t'-equivalent to a t-loop. This one is, by assumption, homotopically trivial in $R_{t'}(X)$. Thus $R_{t'}(X)$ is simply connected. $\square$

From Lemma 4.4, Proposition 4.5 and Proposition 4.8, we get

Corollary 4.9. *Let G be a group generated by a subset S, endowed with the word metric. Then G is boundedly presented by S if and only G is coarsely simply connected. In particular, a locally compact compactly generated group is compactly presented if and only if it is coarsely simply connected.*

Proposition 4.10. *Being coarsely simply connected is a quasi-isometry invariant.*

Proof. Consider metric spaces X, Y, and functions $f : X \to Y$, $g : Y \to X$ such that

$$\forall x, x' \in X,\ d(f(x), f(x')) \leq Ad(x, x') + B\,;$$
$$\forall y, y' \in X,\ d(f(y), f(y')) \leq \alpha d(y, y') + \beta\,;$$
$$\forall x \in X,\ d(g \circ f(x), x) \leq C\,; \quad \forall y \in X,\ d(f \circ g(y), y) \leq \gamma.$$

Suppose that Y is coarsely simply connected.

Then Y is r-connected for some r. If $x, x' \in X$, then there exists a r-path $(f(x) = y_0, \dots, y_n = f(x'))$. Set $r' = \max(C, \alpha r + \beta)$. Then $(x, g \circ f(x) = g(y_0), \dots, g(y_n) = g \circ f(x'), x')$ is a r'-path joining x and x', so that X is r'-connected.

Consider now a ρ-loop $x_0, \dots, x_n = x_0$ in X. Then $f(x_0), \dots, f(x_n)$ is a $(A\rho+B)$-loop in Y. Since Y is coarsely simply connected, there exists $R \geq A\rho+B$ (depending only on $A\rho + B$ and not on the loop) such that $f(x_0), \dots, f(x_n)$ is R-equivalent to the trivial loop. Thus $g \circ f(x_0) \dots g \circ f(x_n)$ is $(\alpha R + \beta)$-equivalent to the trivial loop. Setting $\rho' = C + \max(\alpha R + \beta, \rho)$, we obtain that $(x_0, \dots, x_n)$ is ρ'-equivalent to $g \circ f(x_0) \dots g \circ f(x_n)$, hence is ρ'-equivalent to the trivial loop. $\square$

Corollary 4.11. *Among locally compact compactly generated groups, being compactly presented is a quasi-isometry invariant.*

Given a metric space X and a path $\gamma : [0, 1] \to X$, its diameter is defined as $\sup_{u,v\in[0,1]} d(\gamma(u) - \gamma(v)$. We call a metric space weakly geodesic if there exists a function $w : \mathbf{R}_+ \to \mathbf{R}_+$ such that, whenever $x, y \in X$, they are joined by a path of diameter $\leq w(d(x, y))$.

Proposition 4.12. *Let X be a weakly geodesic metric space. If X is simply connected, then it is coarsely simply connected.*

Proof. We can suppose that $w(r) \geq r$ for all r.

Clearly, X is path-connected. Fix r, and consider a r-loop $(x_0, x_1, \dots, x_n = x_0)$ in X. Interpolate it to obtain a path $\gamma : [0, n] \to X$ such that $\gamma(i) = x_i$ for all $i = 0, \dots, n$, and such that, for all $i \leq t, t' \leq i+1$, we have $d(\gamma(t), \gamma(t')) \leq R = w(r)$.

The path γ is homotopically trivial. Thus there is a continuous function $h : [0, 1]^2 \to X$ such that, for all t, $h(t, 0) = \gamma(t)$ and $h(0, t) = h(1, t) = h(t, 1) = x_0 = x_n$ for all t.

Since h is uniformly continuous, there exists η such that, for all x, y, x', y', if $\max(|x - x'|, |y - y'|) \leq \eta$, then $d(h(x, y), h(x', y')) \leq R$.

Using the assumption on the diameter, the R-path $(x_0, x_1, \dots, x_n)$ is R-homotopic to a path of the form $(\gamma(t_0), \dots, \gamma(t_m)$, with $0 = t_0 \leq t_1 \leq \dots \leq t_n = 1$, and $t_{j+1} - t_j \leq \eta$ for all j.

Now

$$\begin{aligned}(\gamma(t_0), \dots, \gamma(t_m) &= (h(t_0, t_0), h(t_1, t_0), \dots, h(t_n, t_0))\\ &= (h(t_0, t_1), h(t_1, t_0), \dots, h(t_n, t_0))\end{aligned}$$

is R-equivalent to $(h(t_0, t_1), h(t_1, t_1), h(t_2, t_0) \dots, h(t_n, t_0))$ (passing through

$$(h(t_0, t_1), h(t_1, t_1), h(t_1, t_0), h(t_2, t_0) \dots, h(t_n, t_0))\ ).$$

Iterating in a similar way, it is R-equivalent to $(h(t_0, t_1), h(t_1, t_1), \dots, h(t_n, t_1))$. Iterating all the process, we obtain that it is R-equivalent to

$$(h(t_0, t_1), h(t_1, t_1), \dots, h(t_n, t_1)) = (x_0, x_0, \dots, x_0),$$

which is R-equivalent to the trivial loop. □

4.2. **Coverings.**

Definition 4.13. Let $X \to Y$ be a map between metric spaces. We call it a r-metric covering if, for every $y \in Y$, such that the closed r-balls $B'(x,r)$, $x \in f^{-1}(y)$ are pairwise disjoint, and that, for every $x, x' \in X$ such that $d(x,x') \le n$, we have $d(x,x') = d(f(x), f(x'))$ (in particular, f is an isometry in restriction to $(n/2)$-balls).

Proposition 4.14. *Let Y be (r,R)-simply connected. Let X be another metric space and $f : X \to Y$ be a R-covering. Then f is injective.*

Proof. Suppose that $f(x) = f(x') = y_0$ for some $x_0, x_1 \in X$. Consider a r-path $(x = x_0, x_1, \dots, x_n = x')$ between x and x'. Write $y_i = f(x_i)$. Consider a R-combinatorial homotopy between $(y_0, \dots, y_n = y_0)$ and the trivial loop (y_0). Since f is a R-covering, the homotopy lifts in a unique way to X. Thus $x = x'$. □

Proposition 4.15. *Let $1 \to N \to G \to Q \to 1$ be an exact sequence of locally compact groups. Suppose that G is compactly generated and that Q is compactly presented. Then N is compactly generated as a normal subgroup of G.*

Proof. Let S be a generating subset of G, whose interior contains 1, and let B_n denote the n-ball of G relative to S. Let N_n be the normal subgroup of G generated by $G \cap B_n$. We must show that eventually $N = N_n$.

Endow G/N_n with the generating subset S_n, defined as the image of S in G/N_n. Then the projection of G/N_{2n+1} onto G/N is a n-covering. Since G/N is coarsely simply connected, this implies, by Proposition 4.14, that the projection $G/N_n \to G/N$ is injective for sufficiently large n. Thus eventually $N_n = N$. □

5. Applications and examples

Proposition 5.1. *Let G be a locally compact group with finitely many connected components. Then G is compactly presented.*

Proof. Clearly, G is compactly generated. It is known (see [MonZ]) that G has a compact normal subgroup W such that G/K is a Lie group. By a result of Mostow [Most], it follows that G has a maximal compact subgroup K such that G/K is diffeomorphic to a Euclidean space. Taking a G-invariant Riemannian metric on G/K, we obtain that G is quasi-isometric to G/K, which is simply connected and geodesic. By Proposition 4.12, G is coarsely simply connected, hence is compactly presented. □

Remark 5.2. When G is a simply connected Lie group, it also follows from Lemma 3.6 that G is compactly presented. Thus, the (true) fact that every connected Lie group is compactly presented is equivalent to each of the following assertions: (i) Every connected Lie group has a finitely generated fundamental group; (ii) every discrete, central subgroup in a connected Lie group is finitely generated. Note that (i) also follows directly from [Most].

Proposition 5.3. *Let G be a locally compact, compactly generated group of polynomial growth. Then it is compactly presented.*

Proof. By a result of Losert [Lo] (relying on a result of Gromov [Gr81]), G lies in an iterate extension compact-by-(connected Lie group)-by-(discrete virtually nilpotent). Thus G is compactly presented. □

Remark 5.4. It can be shown (see [Bre]) that a locally compact, compactly generated group of polynomial growth is quasi-isometric to a simply connected nilpotent Lie group.

Example 5.5. Let $\mathbf{K}$ be a non-Archimedean local field (e.g. $\mathbf{K} = \mathbf{Q}_p$). Define the Heisenberg group $H_3(\mathbf{K})$ as follows: as a topological space, $H_3(\mathbf{K}) = \mathbf{K}^3$, and its group law is given by $(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1y_2 - x_2y_1)$. The group $\mathrm{SL}_2(\mathbf{K})$ acts on $H_3(\mathbf{K})$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (x, y, z) = (ax + by, cx + dy, z)$. It is easy to check that the semidirect product $\mathrm{SL}_2(\mathbf{K}) \ltimes H_3(\mathbf{K})$ is compactly generated. Its centre coincides with the centre $\{0\} \times \{0\} \times \mathbf{K}$ of $H_3(\mathbf{K})$, hence is not compactly generated as a normal subgroup of $\mathrm{SL}_2(\mathbf{K}) \ltimes H_3(\mathbf{K})$. Thus, by Proposition 4.15, the quotient by the centre, namely $\mathrm{SL}_2(\mathbf{K}) \ltimes \mathbf{K}^2$, is compactly generated but not compactly presented.

Proposition 5.6. *Let G be a Gromov-hyperbolic, compactly generated locally compact group. Then G is compactly presented.*

Proof. Gromov then shows [Gr87, 1.7.A] that the Rips complex $R_t(G)_{\mathrm{top}}$ is contractible for t large enough. So Proposition 4.8 applies. □

Appendix A. Some locally compact groups not quasi-isometric to homogeneous locally finite graphs

We here justify the following well-known result

Proposition A.1. *Let G is a simply connected graded nilpotent Lie group with no lattice (i.e. the Lie algebra has no form over the rationals, see* [Rag, Theorem 2.12]*). Then G is not quasi-isometric to a homogeneous graph.*

An example of a simply connected nilpotent Lie group with no lattice is given in [Rag, Remark 2.14]; the examples given there are nilpotent of class two and therefore are obviously graded.

Proof. Trofimov [Tro] proved, relying on Gromov's polynomial growth theorem [Gr81] that a homogeneous (i.e. vertex-transitive) graph with polynomial growth is quasi-isometric to a finitely generated torsion-free nilpotent group Γ. So G is quasi-isometric to the Malcev closure N of Γ (a simply connected nilpotent Lie group in which Γ is a lattice). Then M having a lattice, it has a form over the rationals, and therefore so does the associated graded nilpotent group M_g. Then by results of Pansu [Pan], M_g is isomorphic to G. We thus get a contradiction. □

Many other Lie groups without cocompact lattices are likely not to be quasi-isometric to any homogeneous graphs, including

- Semidirect products $G_\alpha = \mathbf{R}^2 \rtimes \mathbf{R}$ with the action by diagonal matrices with coefficients $(e^t, e^{\alpha t}$, excluding the special cases $\alpha = -1, 0, 1$. (The case $\alpha > 0$ should be easier to tackle as then G is negatively curved.)
- Semidirect products $\mathbf{R}^n \rtimes \mathrm{GL}(n, \mathbf{R})$ or $\mathbf{R}^n \rtimes \mathrm{SL}(n, \mathbf{R})$ for $n \geq 2$.

Note that in general, a space is quasi-isometric to a homogeneous graph of bounded degree if and only if it is quasi-isometric to a totally discontinuous compactly generated locally compact group: indeed, any such graph is quasi-isometric to its isometry group, and conversely for any totally discontinuous compactly generated locally compact group G, if K is an open compact subgroup then G/K carries a left-invariant bounded degree graph structure (this elementary construction has been known for decades but the older reference I could find is [Mon, Chap. 11, p. 150]).

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