KMS states on C^*-algebras of $*$-commuting local homeomorphisms and applications in k-graph algebras

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Product system of Hilbert bimoduls

Let A be a C^*-algebra. A right Hilbert A–A bimodule is a right A-module X equibed with

(a) An A-valued inner product such that $\langle x, y \cdot a\rangle_A = \langle x, y\rangle_A a$, and X is complete in the norm given by $\|x\| = \|\langle x, x\rangle_A\|^{\frac{1}{2}}$.

(b) A homomorphism $\varphi : A \to \mathcal{L}(X)$. We view φ as a left action of A on X and write $a \cdot x$ for $\varphi(a)(x)$.

- For $x, y \in X$, there is an adjointable operator $\Theta_{x,y}$ on X such that

$$\Theta_{x,y}(z) = x \cdot \langle y, z\rangle$$

The algebra of (compact operators) is

$$\mathcal{K}(X) := \overline{\text{span}}\{\Theta_{x,y} : x, y \in X\} \subset \mathcal{L}(X)$$

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Let P be a semigroup with identity e. A product system over P of right Hilbert A–A bimodule is $X := \bigsqcup_{p \in P} X_p$ such that

(P1) For $p \in P$, X_p is a right Hilbert A–A bimodule.

(P2) The identity fibre X_e equals the standard bimodule ${}_A A_A$.

(P3) X is a semigroup and for each $p, q \in P \setminus \{e\}$ the map $(x, y) \mapsto xy : X_p \times X_q \to X_{pq}$, extends to an isomorphism $\sigma_{p,q} : X_p \otimes_A X_q \to X_{pq}$.

(P4) The multiplications $X_e \times X_p \to X_p$ and $X_p \times X_e \to X_p$ satisfy

$$ax = \varphi_p(a)z, \quad xa = x \cdot a \text{ for } a \in X_e \text{ and } x \in X_p.$$

If P is a subsemigroup of a group G such that $P \cap P^{-1} = \{e\}$. Then $p \leq q \Leftrightarrow p^{-1}q \in P$ defines a partial order on G.

We say (G, P) is a quasi-lattice ordered group if for any two elements $p, q \in G$ which have a common upper bound in P there is a least upper bound $p \vee q \in P$. Let $p \vee q = \infty$ when $p, q \in G$ have no common upper bound.

A product system over P in the quasi-lattice ordered group (G, P) is compactly aligned, if for all $p, q \in P$ with $p \vee q < \infty$, $S \in \mathcal{K}(X_p)$ and $T \in \mathcal{K}(X_q)$, we have

$$(S \otimes_A 1)(T \otimes_A 1) \in \mathcal{K}(X_{p \vee q}).$$

Representations

Let B be a C^*-algebra. A function $\psi : X \to B$ is a (Toeplitz) representation of X if:

(T1) For each $p \in P \setminus \{e\}$, $\psi_p : X_p \to B$ is linear, and $\psi_e : A \to B$ is a homomorphism,

(T2) $\psi_p(x)^*\psi_p(y) = \psi_e(\langle x, y\rangle)$ for $p \in P$, and $x, y \in X_p$, and

(T3) $\psi_{pq}(xy) = \psi_p(x)\psi_q(y)$ for $p, q \in P$, $x \in X_p$, and $y \in X_q$.

The conditions (T1) and (T2) induce a homomorphism $\psi^{(p)} : \mathcal{K}(X_p) \to B$ such that $\psi^{(p)}(\Theta_{x,y}) = \psi_p(x)\psi_p(y)^*$ (see [6]).

Let (G, P) be a quasi-lattice ordered group and let X be a compactly aligned product system over P. A Toeplitz representation ψ of X is Nica-covariant if for every $p, q \in P$, $S \in \mathcal{K}(X_p)$, and $T \in \mathcal{K}(X_q)$, we have

$$\psi^{(p)}(S)\psi^{(q)}(T) = \begin{cases} \psi^{(p \vee q)}\big((S \otimes_A 1)(T \otimes_A 1)\big) & \text{if } p \vee q < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Nica-Toeplitz algebra

Fowler showed in [2, Theorem 6.3] that there exist a C^*-algebra $\mathcal{NT}(X)$ and a Nica-covariant Toeplitz representation ψ of X in $\mathcal{NT}(X)$ such that:

(U) For any other Nica-covariant Toeplitz representation θ of X in a C^*-algebra B, there exists a unique homomorphism $\theta_* : \mathcal{NT}(X) \to B$ such that $\theta_* \circ \psi = \theta$.

- In addition
$\mathcal{NT}(X) = \overline{\text{span}}\{\psi_p(x)\psi_q(y)^* : p, q, n \in P, x \in X_p, y \in X_q\}$.

The Cuntz-Pimsner algebra $\mathcal{O}(X)$ is the quotient of $\mathcal{NT}(X)$ by the ideal

$$\left\{\psi(a) - \psi^{(p)}(\varphi_p(a)) : p \in P, a \in \varphi_p^{-1}(\mathcal{K}(X_p))\right\}.$$

- There is a gauge action $\lambda : \mathbb{T}^k \to \mathrm{Aut}(\mathcal{NT}(X))$ such that $\lambda_z(\psi_m(x)\psi_n(y)^*) = z^{m-n}(\psi_m(x)\psi_n(y)^*)$.
- Fix $r \in R^k$, we can define $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{NT}(X))$ by $\alpha_t = \gamma_{e^{itr}}$ (where $e^{itr} = (e^{itr_1}, \ldots, e^{itr_k})$).
- For each $\psi_m(x)\psi_n(y)^* \in \mathcal{NT}(X)$, the function $t \mapsto \alpha_t\big(\psi_m(x)\psi_n(y)^*\big) = e^{it(m-n)}\psi_m(x)\psi_n(y)^*$ on $\mathbb{R}$ extends to an entire function on all of $\mathbb{C}$.

A product system associated to a family of local homeomorphisms

Let $h_1, \ldots, h_k$ be surjective local homeomorphisms on a compact Hausdorff space Z.

- For $m \in \mathbb{N}^k$ let $h^m := h_1^{m_1} \circ \cdots \circ h_k^{m_k}$, and let $A := C_0(Z)$. There is a right action of A on $C_c(Z)$ and there is a well defined A-valued inner product on $C_c(Z)$ such that

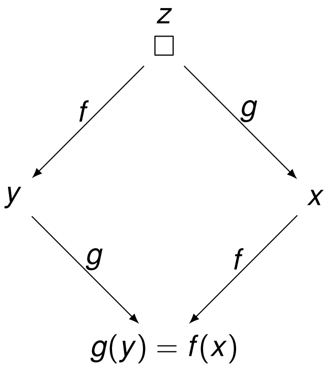
$$(x \cdot a)(z) = x(z)a(h^m(z)), \text{ and}$$
$$\langle x, y \rangle_A(z) = \sum_{h^m(w)=z} \overline{x(w)}y(w).$$

  Let X_m be the completion of $C_c(Z)$ in the arising norm. The formula $(a \cdot x)(z) := a(z)x(z)$ defines a left action of A by adjointable operators on X.
- $X := \bigsqcup_{m \in \mathbb{N}^k} X_m$ is a compactly align product system over $\mathbb{N}^k$ with the multiplication given by

$$xy(z) := x(z)y(h^m(z)) \text{ for } x \in X_m, y \in Y_n, z \in Z$$

$*$-commuting maps

Let f, g be commuting maps on a set Z. We say f and g *$*$-commute* , if for every $x, y \in Z$ satisfying $f(x) = g(y)$, there exists a unique $z \in Z$ such that $x = g(z)$ and $y = f(z)$.



- A family of maps $*$-commute if any two of them $*$-commute.

A characterisation of KMS states

Proposition. Let $h_1, \ldots, h_k$ be $*$-commuting and surjective local homeomorphisms on a compact Hausdorff space Z and let X be the associated product system. Suppose $r \in (0, \infty)^k$ and $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{NT}(X))$ is given in terms of the gauge action by $\alpha_t = \gamma_{e^{itr}}$. Let $\beta > 0$ and ϕ be a state on $\mathcal{NT}(X)$.

(a) If ϕ satisfies

$$\phi\big(\psi_m(x)\psi_n(y)^*\big) = \delta_{m,n} e^{-\beta r \cdot m} \phi \circ \psi_0(\langle y, x\rangle), \qquad (1)$$

then ϕ is a KMS$_\beta$ state of $(\mathcal{NT}(X), \alpha)$.

(b) If ϕ is a KMS$_\beta$ state of $(\mathcal{NT}(X), \alpha)$ and $r \in (0, \infty)^k$ has rationally independent coordinates, then ϕ satisfies (1).

A finite regular Borel measure ν on Z can be viewed as an element of $C(Z)^*$ by

$$\nu(a) := \int a(z)\, d\nu(z) \text{ for } a \in C(Z).$$

We can then calculate a formula for $R^n(\nu)$.

$$\int a\, d\big(R^n(\nu)\big) = \int \sum_{h^n(w)=z} a(w)\, d\nu(z) \quad \text{for } a \in C(Z).$$

We say a measure ν satisfies subinvariance relation if for every subset K of $\{1, \ldots, k\}$, we have

$$\int a\, d\Big(\prod_{i \in K}(1 - e^{-\beta r_i} R^{e_i})\nu\Big) \geq 0 \text{ for all positive } a \in C(Z). \quad (2)$$

Soloutions of the subinvariance relation

Proposition. Let $r \in (0,\infty)^k$ and let

$$\beta_{c_i} := \limsup_{j\to\infty}\left(j^{-1}\ln\left(\max_{z\in Z}|h_i^{-j}(z)|\right)\right).$$

Suppose $\beta \in (0,\infty)$ satisfies $\beta r_i > \beta_{c_i}$. Then

(a) The series $\sum_{n\in\mathbb{N}^k} e^{-\beta r\cdot n}|h^{-n}(z)|$ converges uniformly for $z \in Z$ to a continuous function $f_\beta(z) \geq 1$.

(b) Suppose ε is a finite regular Borel measure on Z. Then the series $\sum_{n\in\mathbb{N}^k} e^{-\beta r\cdot n}R^n\varepsilon$ converges in norm in the dual space $C(Z)^*$ with sum μ, say. Then μ satisfies the subinvariance relation and we have $\varepsilon = \left(\prod_{i=1}^k \left(1 - e^{-\beta r_i}R^{e_i}\right)\right)\mu$. Then μ is a probability measure if and only if $\int f_\beta\, d\varepsilon = 1$.

(c) Suppose μ is a probability measure which satisfies the subinvariance relation. Then $\varepsilon = \left(\prod_{i=1}^k \left(1 - e^{-\beta r_i}R^{e_i}\right)\right)\mu$ is a finite regular Borel measure satisfying $\sum_{n\in\mathbb{N}^k} e^{-\beta r\cdot n}R^n\varepsilon = \mu$, and we have $\int f_\beta\, d\varepsilon = 1$.

Theorem. Suppose $r \in (0,\infty)^k$ satisfies that $\beta r_i > \beta_{c_i}$.

(a) Suppose that ε is a finite regular Borel measure on Z such that $\int f_\beta \, d\varepsilon = 1$, and take $\mu = \sum_{n=0}^{\infty} e^{-\beta n} R^n \varepsilon$. Then there is a KMS_β state ϕ_ε on $(\mathcal{NT}(X),\alpha)$ such that

$$\phi_\varepsilon\big(\psi_m(x)\psi_p(y)^*\big) = \begin{cases} 0 & \text{if } m \neq p \\ e^{-\beta r\cdot m}\int \langle y, x\rangle \, d\mu & \text{if } m = p. \end{cases}$$

(b) If in addition r has rationally independent coordinates, then the map $\varepsilon \mapsto \phi_\varepsilon$ is an affine isomorphism of $\Sigma_\beta := \left\{\varepsilon \in M(Z)_+ : \int f_\beta \, d\varepsilon = 1\right\}$ onto the simplex of KMS_β states of $(\mathcal{NT}(X),\alpha)$.

Proof

- Let $H := \bigoplus_{n\in\mathbb{N}^k} L^2(Z, R^n\varepsilon)$, and define $\theta_m : X_m \to B(H)$ by

$$(\theta_m(x)\xi)_n(z) = \begin{cases} 0 & \text{if } n \not\geq m \\ x(z)\xi_{n-m}(h^m(z)) & \text{if } n \geq k. \end{cases}$$

- θ is a Nica-covariant Toeplitz representation of X. Then there is a homomorphism $\theta_* : \mathcal{NT}(X) \to B(H)$.
- For $q \in \mathbb{N}^k$, choose a partition $\{Z_{q,i} : 1 \leq i \leq l_q\}$ of Z by Borel sets such that h^q is injective on each $Z_{q,i}$. Define $\xi^{q,i} \in H$ by

$$\xi^{q,i}_n = \begin{cases} 0 & \text{if } n \neq q \\ \chi_{Z_{q,i}} & \text{if } n = k. \end{cases}$$

- We aim to define our state $\phi_\varepsilon : \mathcal{NT}(X) \to \mathbb{C}$ by

$$\phi_\varepsilon(b) = \sum_{q\in\mathbb{N}^k} \sum_{i=1}^{l_q} e^{-\beta r\cdot q}\big(\theta_*(b)\xi^{q,i} \,|\, \xi^{q,i}\big) \quad \text{for } b \in \mathcal{T}(X(E)),$$

k-graphs

A k-graph (Λ, d) consists of a countable small category Λ (with range and source maps r and s respectively) together with a functor $d : \Lambda \to \mathbb{N}^k$ satisfying the factorisation property :

for every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there are unique elements $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$ and $d(\mu) = m, d(\nu) = n$.

k-graphs

Suppose that Λ is a k-graph with vertex set Λ^0 and degree map $d : \Lambda \to \mathbb{N}^k$.

- For any $n \in \mathbb{N}^k$, we write $\Lambda^n := \{\lambda \in \Lambda^* : d(\lambda) = n\}$.
- All k-graphs considered here are finite in the sense that Λ^n is finite for all $n \in \mathbb{N}^k$.
- Given $v, w \in \Lambda^0$, $v\Lambda^n w$ denotes $\{\lambda \in \Lambda^n : r(\lambda) = v \text{ and } s(\lambda) = w\}$.
- We say Λ has no sinks if $\Lambda^n v \neq \emptyset$ for every $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.
- Λ has no sources if $v\Lambda^n \neq \emptyset$ for every $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.
- For $\mu, \nu \in \Lambda$, we write

$$\Lambda^{\min}(\mu, \nu) := \{(\xi, \eta) \in \Lambda \times \Lambda : \mu\xi = \nu\eta \text{ and } d(\mu\xi) = d(\mu) \vee d(\nu)\}.$$

k-graphs C^*-algebras

Given a k- graph Λ, a Toeplitz-Cuntz-Krieger Λ-family in a C^*-algebra B is a set of partial isometries $\{S_\lambda : \lambda \in \Lambda\}$ such that

(TCK1) $\{S_v : v \in \Lambda^0\}$ is a set of mutually orthogonal projections,

(TCK2) $S_\lambda S_\mu = S_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$,

(TCK3) $S_\mu^* S_\nu = \sum_{(\xi,\eta)\in\Lambda^{\min}(\mu,\nu)} S_\xi S_\eta^*$ for all $\mu, \nu \in \Lambda$.

We interpret empty sums as 0. We can prove that

$$S_v \geq \sum_{\lambda\in v\Lambda^n} S_\lambda S_\lambda^* \text{ for all } v \in \Lambda^0 \text{ and } n \in \mathbb{N}^k.$$

A Toeplitz-Cuntz-Krieger Λ-family $\{S_\lambda : \lambda \in \Lambda\}$ is a Cuntz-Krieger Λ-family if we also have

(CK) $S_v = \sum_{\lambda\in v\Lambda^n} S_\lambda S_\lambda^*$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

k-graphs C^*-algebras

The Toeplitz algebra $\mathcal{T}C^*(\Lambda)$ is generated by a universal Toeplitz-Cuntz-Krieger Λ-family $\{s_\lambda : \lambda \in \Lambda\}$.

The Cuntz-Krieger algebra $C^*(\Lambda)$ is the quotient of $\mathcal{T}C^*(\Lambda)$ by the ideal

$$\Big\langle s_v - \sum_{\lambda \in v\Lambda^n} s_\lambda s_\lambda^* : v \in \Lambda^0 \Big\rangle.$$

There is a strongly continuous gauge action $\tilde{\gamma} : \mathbb{T}^k \to \mathcal{T}C^*(\Lambda)$ such that $\tilde{\gamma}_z(s_\lambda) = z^{d(\lambda)} s_\lambda$. Since $\tilde{\gamma}$ fixes the kernel of the quotient map, it induces a natural gauge action of $\mathbb{T}^k$ on $C^*(\Lambda)$.

Infinite-path space and shifts

Let $\Omega_k := \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k : m \leq n\}$.

- The set Ω_k is a k-graph with $r(m, n) = (m, m), s(m, n) = (n, n), (m, n)(n, p) = (m, p)$ and $d(m, n) = n - m$.
- The set

  $$\Lambda^\infty := \{z : \Omega_k \to \Lambda : \ z \text{ is a functor intertwining the degree maps}\}$$

  is called infinite-path space of Λ.
- For $p \in \mathbb{N}^k$, the shift map $\sigma^p : \Lambda^\infty \to \Lambda^\infty$ is defined by $\sigma^p(z)(m, n) = z(m + p, n + p)$ for all $z \in \Lambda^\infty$ and $(m, n) \in \Omega_k$.
- Clearly $\sigma^p \circ \sigma^q = \sigma^q \circ \sigma^p$ for $p, q \in \mathbb{N}^k$.

A k-graph Λ is 1-coaligned if for all $1 \leq i \neq j \leq k$ and $(\lambda, \mu) \in \Lambda^{e_i} \times \Lambda^{e_j}$ with $s(\lambda) = s(\mu)$ there exists a unique pair $(\eta, \zeta) \in \Lambda^{e_j} \times \Lambda^{e_i}$ such that $\eta\lambda = \zeta\mu$.

Lemma. Let Λ be a finite 1-coaligned k-graph. Suppose that $0 \leq i \neq j \leq k$. Then the shift maps σ^{e_i} and σ^{e_j} $*$-commute.

For Λ, shifts gives a product system $X(\Lambda^\infty)$. We write $\mathcal{NT}(X(\Lambda^\infty))$ and $\mathcal{O}(X(\Lambda^\infty))$ for the corresponding Nica-Toeplitz algebra and Cuntz-Pimsner algebra.

Proposition. Let Λ be a finite 1-coaligned k-graph with no sinks or sources. For each $\lambda \in \Lambda$, let $S_\lambda := \psi_{d(\lambda)}(\chi_{Z(\lambda)})$. Then

(a) The set $\{S_\lambda : \lambda \in \Lambda\}$ is a Toeplitz-Cuntz-Krieger Λ-family in $\mathcal{NT}(X(\Lambda^\infty))$. The homomorphism $\pi_S : \mathcal{T}C^*(\Lambda) \to \mathcal{NT}(X(\Lambda^\infty))$ is injective and intertwines the respective gauge actions of $\mathbb{T}^k$ (that is, $\pi_S \circ \tilde{\gamma} = \gamma \circ \pi_S$).

(b) Let $q : \mathcal{NT}(X(\Lambda^\infty)) \to \mathcal{O}(X(\Lambda^\infty))$ be the quotient map. Then $\{q \circ S_\lambda : \lambda \in \Lambda\}$ is a Cuntz-Krieger Λ-family in $\mathcal{O}(X(\Lambda^\infty))$. The corresponding homomorphism $\pi_{q\circ S} : C^*(\Lambda) \to \mathcal{O}(X(\Lambda^\infty))$ is an isomorphism and intertwines the respective gauge actions of $\mathbb{T}^k$.

Theorem 6.1 [aHLRS-2014]. Let Λ be a finite k-graph without sources, and let A_i be the vertex matrices of Λ. Suppose that $r \in (0, \infty)^k$ satisfies $\beta r_i > \ln \rho(A_i)$ for $1 \leq i \leq k$, and define $\tilde{\alpha} : \mathbb{R} \to \mathrm{Aut}(\mathcal{T}C^*(\Lambda)$ by$\tilde{\alpha}_t = \tilde{\gamma}_{e^{itr}}$
For $v \in \Lambda^0$, the series $\sum_{\mu \in v\Lambda} e^{-\beta r \dot{d}(\mu)}$ converges with sum $y_v \geq 1$. Set $y = (y_v) \in [1, \infty)^{\Lambda^0}$ Then there is an affine issomorphism from

$$\Sigma_\beta := \{\epsilon \in [0, \infty)^{\Lambda^0} : \epsilon \cdot y = 1\}$$

onto the simplex of KMS$_\beta$ states of $(\mathcal{T}C^*(\Lambda), \tilde{\alpha})$.

Corollary. The injection $\pi_S : \mathcal{T}C^*(\Lambda) \to \mathcal{NT}(X(\Lambda^\infty))$ is not a surjection and $\mathcal{T}C^*(\Lambda)$ is substantially smaller than $\mathcal{NT}(X(\Lambda^\infty))$.

Z. Afsar, A. an Huef and I. Raeburn, KMS states on C^*-algebras associated to local homeomorphisms, *Internat. J. Math.* **25** (2014), no. 8, 1450066(27 pages).

N. J. Fowler, Discrete product systems of Hilbert bimodules, *Pacific J. Math.* **204** (2002), 335–375.

A. an Huef, M. Laca, I. Raeburn and A. Sims, KMS states on the C^*-algebras associated to higher-rank graphs, *J. Funct. Anal.* **266** (2014), 265–283.

A. Kumjian and D. Pask, Higher-rank graph C^*-algebras, *New York J. Math.* **6** (2000), 1–20.

M. Laca and S. Neshveyev, KMS states of quasi-free dynamics on Pimsner algebras, *J. Funct. Anal.* **211** (2004), 457–482.

M.V. Pimsner, A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by Z, *Fields Institute Comm.* **12** (1997), 189–212.