Classification of non-singular systems and critical dimension

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1 Introduction

Let $(X, \mathcal{B}, \mu, T)$ be a measurable dynamical system. where $\mathcal{B}$ is a σ-algebra of sets and μ is a measure (often, for me, a probability measure), equipped with a bimeasurable transformation $T : X \to X$. The system is **measure-preserving** if $\mu \circ T = \mu$, but I will mostly be discussing the case when μ is **non-singular** ie $\mu \circ T \sim \mu$.

For non-singular systems, we must deal with the Radon-Nikodým derivatives $\omega_k(x) := \frac{d\mu \circ T^k}{d\mu}(x)$.

We shall assume also that the system is **ergodic**: every invariant set of positive measure must have complement of measure zero.

Two non-singular dynamical systems $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ are said to be **metrically isomorphic** or **conjugate** if there exists a bimeasurable invertible mapping $\Phi : X \to Y$ such that $\Phi \circ T = S \circ \Phi$ and $T \circ \Phi^{-1} = \Phi^{-1} \circ S$ a.e. and $\nu \circ \Phi^{-1} \sim \mu$.

They are **orbit equivalent** if there exists Φ as above with for all n and a.e. x, $\Phi \circ T^n x = S^{\sigma(x,n)} \circ \Phi x$.

I'll talk more about the interesting function $\sigma(x, n)$ later. It is is an integer-valued cocycle.

Von Neumann (1930's) proposed an initial classification of ergodic non-singular dynamical systems into:

- Type I_n systems: $n = 1, 2, \ldots, \infty$; discrete systems with X finite or countably infinite
- Type II_1 systems: where there is a finite measure which is preserved by T. (We can assume it's a probability space.)
- Type II_∞ systems: where there is an infinite measure which is preserved by T.
- Type III systems: where there is no measure preserved by T.

Henry Dye (1959) showed that every non-singular dynamical system of type II or III is orbit equivalent to an infinite product system $X = \prod_{i=1}^{\infty}\{0, 1\}$, T is the odometer, but with not much information over the measure. (For type II_1 it's $\otimes_{i=1}^{\infty}\mu_i$ where all the μ_i are the $\frac{1}{2}$-$\frac{1}{2}$ measure on $\{0, 1\}$.)

In the 1970's Krieger introduced the ratio set: it is a subgroup of $\mathbb{R}_+$ which consists of the "essential " values of $\frac{d\mu\circ T^n}{d\mu}(x)$. This allowed one to further refine the type III systems into III_λ for $0 \leq \lambda \leq 1$. Connes and Kreiger showed that the III_1 and III_λ for $\lambda > 0$ systems are unique up to *orbit equivalence.*

Connes and Krieger showed that Type III_0 are classified up to orbit equivalence by their associated flows (up to conjugacy). However, there are no explicit parameterisations of them (and this may be impossible.)

Theorem 1 *(D. and Hamachi) Every ergodic non-singular dynamical system* $(X, \mathcal{B}, T, \mu)$ *is orbit equivalent to a Markov odometer on a Bratteli-Vershik diagram. Furthermore, when considered as a* G*-measure, the Markov odometer may be taken to be:*

- *uniquely ergodic*
- *a minimal transformation for the topology of* X
- *an induced transformation of a full odometer*

This is an extension of the Dye theorem and the Jewett-Krieger Theorem.

Hamachi and I also gave an explicit construction of a Markov odometer which is not orbit equivalent to a product measure, but where the number of vertices grows extremely quickly. The rate of growth is somehow a crucial ingredient in measuring the complexity of the system.

Question: Is there some quantity like entropy which enables us to make a finer classification of orbit equivalence classes?

2 Measure-preserving systems and entropy

Much of the ergodic theory literature has been devoted to the measure-preserving case, and the greater part of the results described below have been extended where one replaces the powers of a single transformation T by an amenable group of transformations which preserves the measure.

Shannon introduced the notion of entropy in information theory, and Kolmogorov and Sinai extended his work in the study of measure-preserving dynamics, introducing the *entropy* of the transformation T: in some senses this is the study of the ergodic theory of $X = \prod_{i=-\infty}^{\infty}\{0, 1\}$ with the *shift* operation.

First, choose a partition $\mathcal{P}$ of X. Its entropy is

$$H(\mathcal{P}) = -\sum_{P\in\mathcal{P}} \mu(P) \log \mu(P).$$

Now note that the limit $\frac{1}{n}H(\mathcal{P} \vee T\mathcal{P} \vee T^2\mathcal{P} \cdots \vee T^n\mathcal{P})$ exists. Call it $H(T, \mathcal{P})$. Taking the largest entropy over all partitions $\mathcal{P}$ we get the entropy $H(T)$. This is alternatively calculated by taking $H(T, \mathcal{P})$ for a generating partition.

Entropy is an invariant of metric equivalence. This enabled Kolmogorov and Sinai to show that the 2-shift isn't equivalent to the 3-shift.

Ornstein and Shields showed that for *Bernoulli systems*, entropy is a complete invariant: ie Bernoulli systems of the same entropy are metrically isomorphic.

If the hypothesis that the system is Bernoulli is dropped, the result goes badly awry: there are uncountable many cpe systems of any given entropy, pairwise non-isomorphic. Golodets, Rudolph, Sinelshchikov and I generalised this result to amenable groups which have an element of infinite order.

In the case of non-singular systems where μ is not preserved, the entropy limit may not exist, or may be zero.

3 Critical dimension and entropy

Let $(X, \mathcal{B}, \mu, T)$ be a non-singular conservative ergodic dynamical system with $\mu(X) = 1$. Let

$$X_{\alpha'} = \{x \in X : \liminf_{n\to\infty} \frac{\sum_{i=0}^{n-1} \omega_i(x)}{n^{\alpha'}} > 0\},$$

and notice that $X_{\alpha'}$ is an invariant set. The supremum over the set of α' for which $\mu(X_{\alpha'}) = 1$ is called the **lower critical dimension** α of $(X, \mathcal{B}, \mu, T)$.

Let

$$X_{\beta'} = \{x \in X : \limsup_{n\to\infty} \frac{\sum_{i=1}^{n} \omega_i(x)}{n^{\beta'}} = 0\}.$$

Let β be the infimum of the set $\{\beta' : \mu(X'_\beta) = 1\}$, the **upper critical dimension**.

Theorem 2 *(D. and Mortiss) The upper and lower critical dimensions are invariants for metric isomorphism.*

Proof is a simple consequence of the Hurewicz ergodic theorem.

If μ is defined on an infinite product space X, we define the **upper** and **lower average coordinate entropy** of μ:

$$\overline{h}_{AC} = \limsup_{n\to\infty} \frac{1}{\log(s(n))} \sum_{i=0}^{n-1} H(\mu_i) \tag{1}$$

and

$$\underline{h}_{AC} = \liminf_{n\to\infty} \frac{1}{\log(s(n))} \sum_{i=0}^{n-1} H(\mu_i)$$

where $H(\mu_i) = -\sum_{j=0}^{l(i)-1} \mu_i(\{j\}) \log \mu_i(\{j\})$ is the usual entropy of the ith coordinate measure, and $s(n)$ is the number of cylinders of length n, ie $s(n) = l(1) \dots l(n)$.

Theorem 3 *(D. and Mortiss) For the odometer action T on $(\prod_{i=1}^{\infty} \mathbb{Z}_{l(i)}, \otimes_{i=1}^{\infty} \mu_i)$ and for $2 \leq l(i) \leq m < \infty$, the lower critical dimension is given by the formula*

$$\alpha = \liminf_{n\to\infty} -\frac{\sum_{i=1}^{n} - \log(\mu_i(x_i))}{\log(s(n))} = \underline{h}_{AC}(\mu),$$

for μ-almost every x.

The upper critical dimension is given by

$$\beta = \limsup_{n\to\infty} -\frac{\sum_{i=1}^{n} - \log(\mu_i(x_i))}{\log s(n)} = \overline{h}_{AC}(\mu)$$

for μ almost all $x \in X$.

In work with Rika Hagihara, we showed that the analogous theorem holds for Markov odometers.

If the AC entropy limit exists then the upper and lower critical dimensions of the associated product odometer actions are equal. In this case, we will refer to either as the **critical dimension**. In this case, we also have

$$\lim_{n\to\infty} -\frac{\sum_{i=1}^{n} \log \mu_i(x_i)}{\log(s(n))} = \alpha, \quad \text{a.e.}$$

which is an analogue of the **Shannon-MacMillan-Breiman Theorem** for Bernoulli shifts.

4 Hurewicz maps

Suppose that $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ are orbit equivalent, i.e. there exists an invertible, bi-measurable mapping $\Phi : X \to Y$ such that $\nu \circ \Phi \sim \mu$, and a cocycle $\sigma : \mathbb{Z} \times X \to \mathbb{Z}$ such that

$$S^n \Phi(x) = \Phi T^{\sigma(n,x)}(x).$$

The cocycle σ encodes information about Φ, together with μ: it allows us to decide when two orbit equivalent systems have the same critical dimension. We know that for orbit equivalence classes of type II_∞ or type III_λ, any value of critical dimension between 0 and 1 is possible. It is interesting to define a notion of equivalence which preserves the critical dimension.

Theorem 4 *Suppose that for almost all $x \in X$*

$$0 < \liminf_n \frac{\sum_{i=0}^n \omega_i(x)}{\sum_{i=0}^n \omega_{\sigma(i,x)}(x)} \leq \limsup_n \frac{\sum_{i=0}^n \omega_i(x)}{\sum_{i=0}^n \omega_{\sigma(i,x)}(x)} < \infty.$$

Then the upper and lower critical dimensions of $(X, \mathcal{B}, \mu, T)$ *and* $(Y, \mathcal{C}, \nu, S)$ *agree.*

Such a Φ is called an **Hurewicz map**.

Two systems are said to be **Hurewicz equivalent** if there exists an orbit equivalence Φ between them which is Hurewicz and such that Φ^{-1} is also Hurewicz.

Recently, my student Daniel Mansfield and I proved the following generalisation of my theorem with Hamachi:

Theorem 5 *Any non-singular ergodic dynamical system is Hurewicz equivalent to a BV system with a Markov measure, and the latter can be taken to be uniquely ergodic and the system minimal and induced from an odometer.*

To show this, it was necessary to show that the induced transformation of a Markov odometer on a set of positive measure is Hurewicz equivalent to the original transformation. This is reasonably easy to show for cylinder sets, and then we can use the fact that each set of positive measure contains a cylinder of positive measure....

For BV systems of finite width, we have:

Theorem 6 *Two BV systems of finite width are Hurewicz equivalent if and only if they have the same upper and lower critical dimensions.*

This shows that the two critical dimensions characterise finite width BV systems up to Hurewicz equivalence.

5 Examples of Hurewicz maps

Which kinds of orbit equivalences of a system are Hurewicz? They include Mortiss' IC equivalences, infinite permutations of the integers (in the ℓ-point odometer case), etc. Indeed, let Γ_n be the "finite coordinate change " up to the nth level of a Bratteli-Vershik system. Let $[\Gamma_n]$ be the set of maps Ψ so that for all $x \in X$ there exists $\gamma \in \Gamma_n$ such that $\Psi(x) = \gamma.x$. The **finitary normaliser** is the set of orbit equivalence maps $\Phi : X \to X$ such that for all n there exists $m > n$ so that $\Phi \circ [\Gamma_n] \circ \Phi^{-1} \subseteq [\Gamma_m]$.

Theorem 7 *Every element of the finitary normaliser is a Hurewicz map.*

6 The critical dimensions for G-measures

Recent work with Daniel Mansfield studies Riesz products and the related construction of G-measures. Let $X = \Pi_{i=1}^{\infty} \mathbb{Z}_{\ell(i)}$ be the infinite product space, where $\ell(i)$ is some sequence of integers ≥ 2. Let Γ_n be the group of coordinate changes of the first n coordinates, and T the odometer, which has the same orbits as $\Gamma = \bigcup_n \Gamma_n$.

The Radon-Nikodým derivatives of a T quasi-invariant measure can be described by a sequence of functions $G_n = \frac{d\mu}{d\mu^n}$. (Here $\mu^n = \frac{1}{|\Gamma_n|} \Sigma_{\gamma \in \Gamma_n} \mu \circ \gamma$ is the tail measure.)

If $\gamma \in \Gamma_n$ then

$$\frac{d\mu \circ \gamma}{d\mu} = \frac{G_n(\gamma x)}{G_n(x)}.$$

Pollicot and Fan defined the upper and lower pointwise dimension of a measure μ to be

$$\underline{d}_\mu(x) = \liminf_{r\to 0} \frac{\log(\mu(B(x,r)))}{\log(r)}, \ \overline{d}_\mu(x) = \limsup_{r\to 0} \frac{\log(\mu(B(x,r)))}{\log(r)}.$$

They show that these are a.e. independent of x for μ ergodic and that $\underline{d} \leq \dim_H(\mu) \leq \overline{d}$, where $\dim_H(\mu)$ isthe Hausdorff dimension of μ.

We can show that

Theorem 8 *(Dooley-Mansfield) Let μ be a G-measure. Then*

$$\alpha \leq \underline{d} \leq \overline{d} \leq \beta$$

and hence if $\alpha = \beta$ coincide, then they are equal to the Hausdorff dimension of μ.

7 The critical dimension for amenable group actions

Ornstein and Weiss showed that a measure-preserving action of any amenable group is orbit equivalent to an integer action. It's true for non-singular actions as well. (Katznelson and Weiss?)

But what about critical dimension?

My student Kieran Jarrett, has been studying the critical dimension for amenable groups. I'd like to conclude by telling you something of what we have done together.

Firstly, let me point out that, for the integers, the proof that the critical dimension is an invariant of metric isomorphism makes essential use of the Hurewicz ergodic theorem. Actually, we don't have a version of the Hurewicz ergodic theorem for all amenable groups (although, for measure-preserving actions, a version of the Birkhoff ergodic theorem has been proved by Ornstein and Weiss and by Lindenstrauss.)

If G is a countable group acting by non-singular transformations on a space (X, μ), we choose a sequence $B_1 \subseteq B_2 \subseteq \ldots$ of finite subsets of G, which we refer to as a **summing sequence**. Let

$$L_t = \{x \in X : \liminf_{n\to\infty} \frac{1}{|B_n|^t} \Sigma_{g\in B_n} \omega_g(x) > 0\}$$

$$U_t = \{x \in X : \limsup_{n\to\infty} \frac{1}{|B_n|^t} \Sigma_{g\in B_n} \omega_g(x) = 0\}$$

and define the upper and lower critical dimension relative to $\{B_n\}$ by

$$\alpha = \sup\{t : \mu(L_t) = 1\}$$
$$\beta = \inf\{t : \mu(U_t) = 1\}.$$

Question: how does the critical dimension depend upon B_n and on the action? and is it an invariant of metric isomorphism?

In order to prove the answer to the second question, we need to ask whether there is a Hurewicz ergodic theorem for the summing sequence B_n. That is, for $\phi \in L^1(X, \mu,$ let $\hat{g}\phi(x) = \phi(gx)\omega_g(x)$, and ask whether

$$\lim_{n\to\infty} \frac{\Sigma_{g\in B_n} \hat{g}\phi}{\Sigma_{g\in B_n} \hat{g}1} = \int \phi d\mu?$$

If such a theorem holds, then the upper and lower critical dimensions are invariant for metric isomorphisms.

We have considered this theorem for the groups $\mathbb{Z}^n$, the Heisenberg group H_n and the lamplighter group.

A basic ingredient of the proof is the non-singular Følner condition:

A summing sequence B_n satisfies (nsFC) if for all $\sigma \in G$

$$\frac{\Sigma_{g \in B_k \Delta \sigma B_k} \omega_g}{\Sigma_{g \in B_k} \omega_g} \to 0 \ \ a.e.$$

8 The case of $\mathbb{Z}^n$

If we assume that G is contained in metric group $\tilde{G}, \tilde{d}$, where the metric is voidless, then (nsFC) holds provided that, given $\sigma \in G$,

$$\frac{\sum_{g \in \delta_t B_k} \omega_g}{\sum_{g \in B_k \omega_g}} \to 0 \quad a.e.$$

where $t = d(\sigma, 0)$ and δ_t is the thickened boundary, then (nsFC) holds.

We can show this for $\mathbb{R}^n$.

In fact, if (G, d) is well spaced, voidless, the B_n have the multiplicative doubling property and (G, d) has finite intersection dimension dimension, then the ergodic theorem holds.

$\mathbb{R}^n$ has all these properties, if B_n comes from the balls of a metric.

A typical action of $\mathbb{Z}^n$ is a product action of each factor of $\mathbb{Z}$ on a measure space X_i, μ_i for $i = 1, \dots n$. We can define a summing sequence for $\mathbb{Z}^n$ by taking rectangles of different side lengths in each of the factors. (This is also a Følner sequence.) Supposing that the *ith* factor has critical dimensions α_i and β_i, then by choosing different ratios of the side lengths, we can obtain as a critical dimension any convex linear combination of the individual critical dimensions:

$\Sigma_{i=1}^n c_i \alpha_i$ and $\Sigma_{i=1}^n c_i \beta_i$, where $\Sigma_{i=1}^n c_i = 1$.

Proposition 9 *If $\mathbb{Z}^2$ acts on a product space $(Y \times Z, \rho)$, such that the product action is ergodic, then $\rho \sim \mu_1 \times \mu_2$, where μ_1 is its projection on Y and μ_2 its projection on Z*

Therefore this proposition extends to product actions of $\mathbb{Z}^n$.

9 The Heisenberg Group

The integer Heisenberg group $H_1(\mathbb{Z})$ consists of the upper triangular matrices: $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ where $x, y, z \in \mathbb{Z}$.

The n-dimensional Heisenberg group is obtained by replacing x and y by elements of $\mathbb{Z}^n$.

It's one of the simplest non-abelian amenable groups, and of course it is a subgroup of the Heisenberg group $H_n(\mathbb{R})$, where $x, y \in \mathbb{R}^n$ and $z \in \mathbb{R}$.

We have proved an analogue of the Hurewicz theorem for the groups $H_n(\mathbb{Z})$.

The proof follows somewhat the path for $\mathbb{Z}^n$ above, that is we consider $H_n(\mathbb{Z}) \subset H_n(\mathbb{R})$ and use the geometry of $H_n(\mathbb{R})$.

The problem is to find a suitable summing set. Our first thought was to use the Korányi distance or the Carnot-Carathéodory metric, which are usually used for harmonic analysis on the Heisenberg group. However, Mike Hochman has observed that the balls from these metrics fail to have the Besicovitch Covering Property, and he has shown that the summing sets from these metrics do NOT have a Hurewicz type ergodic theorem.

However, recently Le Donne and Rigot (2017) found a metric for which the balls DO satisfy the BCP. Using the balls (B_n) from this metric, we can prove the Hurewicz theorem:

$$\lim_{n\to\infty} \frac{\Sigma_{g\in B_n}\hat{g}\phi}{\Sigma_{g\in B_n}\hat{g}1} = \int \phi d\mu.$$

The Le Donne and Rigot metric is defined as follows: Consider the natural dilation δ_λ on the Heisenberg group $(x, y, z) \to (\lambda x, \lambda y, \lambda^2 z)$. Define the distance between two matrices p, q by:

$$d(p, q) = \inf_{r>0}\{\delta_{1/r}(pq^{-1}) \in B_{eucl})\}$$

where B_{eucl} is the Euclidean ball of radius 1 on $\mathbb{R}^{2n} \times \mathbb{R}$.

Corollary 10 *The critical dimension for the Le Donne-Rigot balls is an invariant of metric isomorphism for* $H_n(\mathbb{Z})$*.*

It follows that the Heisenberg odometers defined by Danilenko have critical dimensions calculated by average coordinate entropy.

10 Current stuff

The multiplicative ergodic theorem of Oseledeč says that if T is a measure-preserving transformation of X and

$$\theta : \mathbb{N} \times X \to GL(r, \mathbb{R})$$

is a cocycle, satisfying $\theta_{m+n} = \theta_m(T^n x)\theta_n(x)$, then the sequence $A(n, x) := (\theta_n(x)^* \theta_n(n))^{\frac{1}{2n}}$ converges almost everywhere to a matrix $A(x) \in M_r(\mathbb{R})$.

It can be used to study products of random matrices and to set up random dynamical systems.

My student Jie Jin is working on extending this theorem to non-singular systems.

Recently he has done the first step, which is to prove a version of the Kingman subadditive ergodic theorem for non-singular systems.