The automorphism groups of the easiest nonabelian infinite groups still present many mysteries

Adam Piggott

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The easiest nonabelian infinite groups...

The free products of easy groups provide the easiest examples of nonabelian infinite groups. Surely no families are easier to understand than the free groups

$$F_n := \langle x_1, \ldots, x_n \mid \qquad \rangle$$

and the universal Coxeter groups

$$W_n := \langle a_1, \ldots, a_n \mid a_1^2 = \cdots = a_n^2 = 1 \rangle.$$

have interesting automorphism groups

The families $\{\mathrm{Aut}(F_n)\}$ and $\{\mathrm{Out}(F_n)\}$ are objects of "classical study" in group theory. They exhibit rich and complex features and their study is valued for its deep connections to geometry and topology.

Although the families$\{\mathrm{Aut}(W_n)\}$ and $\{\mathrm{Out}(W_n)\}$ are less complicated than their celebrated cousins, they present mysteries worth exploring too. In this presentation we will highlight some of the work that has already been done on these families, and some of the questions that remain for you to answer.

Structural results

Each permutation of the set $\{a_1, \ldots, a_n\}$ determines an automorphism of W_n. These automorphisms generate a finite subgroup S_n of $\mathrm{Aut}(W_n)$.

For an integer i and a subset j, the map

$$a_j \mapsto a_i a_j a_i, \quad a_k \mapsto a_k \text{ for } k \neq j$$

determines an automorphism α_{ij} of W_n called the *partial conjugation* with acting letter i and domain j. These automorphisms generate a normal subgroup $\Sigma\mathrm{Aut}(W_n)$ of $\mathrm{Aut}(W_n)$.

It is easy to see that

$$\mathrm{Aut}(W_n) \cong \Sigma\mathrm{Aut}(W_n) \rtimes S_n$$

We can tease out more detail (Gutierrez-P.-Ruane, 2012 [2]):

Let $\Sigma\mathrm{Out}(W_n) = \langle \alpha_{12}, \alpha_{21}, \alpha_{31}, \ldots, \alpha_{n1} \rangle$. Then

$$\mathrm{Aut}(W_n) \cong \underbrace{(W_n \rtimes \Sigma\mathrm{Out}(W_n))}_{\Sigma\mathrm{Aut}(W_n)} \rtimes S_n$$

Connections with other groups

Connections with other interesting families of groups make $\mathrm{Aut}(W_n)$ interesting. They inspire questions, and supply tools. The connections include:

- $\mathrm{Aut}(F_2) \cong \mathrm{Aut}(W_3) \cong \mathrm{Aut}(B_4)$
- $\mathrm{Aut}(W_n) \hookrightarrow \mathrm{Aut}(F_{n-1})$
- $\Sigma\mathrm{Aut}(F_n)$ and $\Sigma\mathrm{Aut}(W_n)$ have very similar presentations
- The stabiliser of a_n in $\mathrm{Aut}(W_n)$ is isomorphic to the group of palindromic automorphisms of F_{n-1}
- W_n is an easy example of a right-angled Coxeter group and F_n is an easy example of a right-angled Artin group.

CAT(0) ?

Gersten proved that $\mathrm{Aut}(F_n)$ is not a CAT(0) group for $n \geq 3$. He did this by exhibiting a poison subgroup that lives in $\mathrm{Aut}(F_n)$. The image of $\mathrm{Aut}(W_n)$ in $\mathrm{Aut}(F_{n-1})$ does not include Gersten's "poison subgroup."

Is $\mathrm{Aut}(W_n)$ a CAT(0) group?

Some results to pique your interest

Theorem (P.-Ruane-Walsh, 2010 [7])

$\mathrm{Aut}(F_2)$ *is a biautomatic CAT(0) group.*

IDEA OF PROOF:

- $\mathrm{Aut}(B_4) \cong B_4/Z(B_4) \rtimes \langle \tau \rangle$
- Brady showed that $B_4/Z(B_4)$ acts faithfully and geometrically on a CAT(0) 2-complex X_0
- We found an extra isometry of X_0 to play the role of the involution τ.

Still open

Is $\mathrm{Aut}(W_n)$ a CAT(0) group for $n \geq 4$?

Is $\mathrm{Out}(W_n)$ a CAT(0) group for $n \geq 4$?

Answering requires finding a poison subgroup, or exhibiting a property of the group incompatible with being a CAT(0) group, or showing that a candidate construction is CAT(0).

A candidate construction

Inspired by (or, at least, by analogy with) Culler and Vogtmann's Outer Space for $\mathrm{Out}(F_n)$, McCullough and Miller (1996) built a simplicial complex associated to the automorphism group of a free products of groups [4].

K_n is constructed by gluing together pieces, each piece a copy of the geometric realisation of the hypertree poset. The gluing instructions use a relationship between commuting products of partial conjugations and hypertrees.

Why hypertrees?

Theorem (P., 2012 [5]*)

For $n \geq 3$, the appropriate McCullough-Miller space K_n is a perfect model for $\mathrm{Out}(W_n)$ in the sense that $\mathrm{Aut}(K_n) \cong \mathrm{Out}(W_n)$.

* This work was completed while on sabbatical at the University of Newcastle.

Accurate geometric models

Group	Accurate geometric model	Credits
Algebraic group (satisfying certain hypothesis)	Spherical building	Tits
Mapping class group associated to a surface of genus at least two	Complex of curves	Royden, Ivanov
Outer automorphisms of F_n for $n \geq 3$	Spine of outer space	Bridson-Vogtmann
Outer automorphisms of W_n for $n \geq 4$	McCullough-Miller space	

IDEA OF PROOF:

- show that the symmetric group of order n is exactly the automorphism group of the simplicial realisation of the hypertree poset
- show that the pieces of K_n intersect in such a way that the position and orientation of the image of one piece completely determines the image of an automorphism.

Some results to pique your interest

Theorem (Cunningham, 2019 [1])

For $n \geq 4$, K_n cannot be given an equivariant CAT(0) metric.

Some results to pique your interest

Theorem (Healy, 2020 [3])

For $n \geq 3$:

- $\mathrm{Out}(W_n)$ *is acylindrically hyperbolic;*
- *If* $\mathrm{Out}(W_n)$ *acts geometrically on a CAT(0) space* X*, then* X *contains a rank-one geodesic;*
- $\mathrm{Out}(W_n)$ *cannot act geometrically on a Euclidean building.*

Another question to pique your interest

We know that the $\mathrm{Aut}(W_3)$ satisfies a quadratic isoperimetric inequality, and $\mathrm{Out}(W_3)$ a linear one (because it is hyperbolic).

What can we say about the isoperimetric inequalities satisfied by $\mathrm{Aut}(W_n)$ and $\mathrm{Out}(W_n)$ for larger values of n?

Another question to pique your interest

A finitely presented group G is automatic if there exists a regular language of normal forms with the fellow-traveller property. Automatic groups satisfy a quadratic isoperimetric inequality, and have many nice algorithmic properties.

We know that $\mathrm{Aut}(W_3)$ and $\mathrm{Out}(W_3)$ are automatic. Are the groups $\mathrm{Aut}(W_n)$ and $\mathrm{Out}(W_n)$ automatic for larger values of n?

Some results to pique your interest

Theorem (P.-Ruane, 2010 [6])

For $n \geq 3$, $\mathrm{Aut}(W_n)$, $\mathrm{Out}(W_n)$, and $\Pi\mathrm{Aut}(F_{n-1})$ admit a regular (in fact, Markov) language of normal forms.

...but these languages are not part of automatic structures.

More results

The following are special cases of more general results

- (Saikat Das, 2018) The group $\mathrm{Out}(W_n)$ is thick, and therefore not relatively hyperbolic, for $n \geq 4$.
- (Lyman, 2019) If ϕ is a polynomially-growing element of $\mathrm{Out}(W_n)$, then the mapping torus $W_n \rtimes \mathbb{Z}$ is a CAT(0) group.

References

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