Coxeter systems for which the Brink-Howlett automaton is minimal.

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Coxeter Systems

- Recall: A **Coxeter System** is a pair (W, S) consisting of a group W and a set of generators $S \subset W$ subject only to relations of the form

$$(st)^{m(s,t)} = 1$$

where $m(s, s) = 1$ and $m(t, s) = m(s, t) \geq 2$ for $s \neq t$. ($m(s, t) = \infty$ is allowed).

Examples

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EXAMPLES

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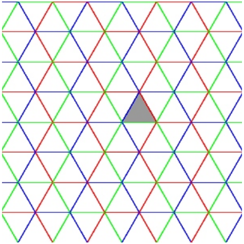
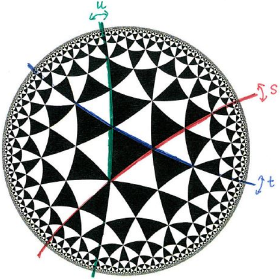
- Weyl groups of simple Lie algebras
- Triangle groups corresponding to tessellations of the Euclidean/Hyperbolic plane.





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  Since $sts = tst$ and $utu = tut$. We have

  $$stsutu = tsttut = tsut$$

COXETER GRAPHS

Coxeter graph Γ of (W, S): vertices labelled by $s \in S$ and there is an edge between vertices s and t if and only if $m(s,t) \geq 3$. The edge is labelled only if $m(s,t) > 3$.

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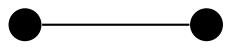

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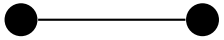

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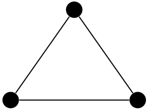

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- Given a string of generators, is it a reduced expression?

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Definition

Let W be a group with generating set S. A ***Finite State Automaton*** *for (W, S) is a finite directed graph capable of reading words $w \in W$ and giving the answer YES if and only if the word w is reduced.*

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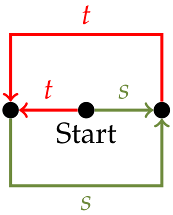
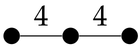


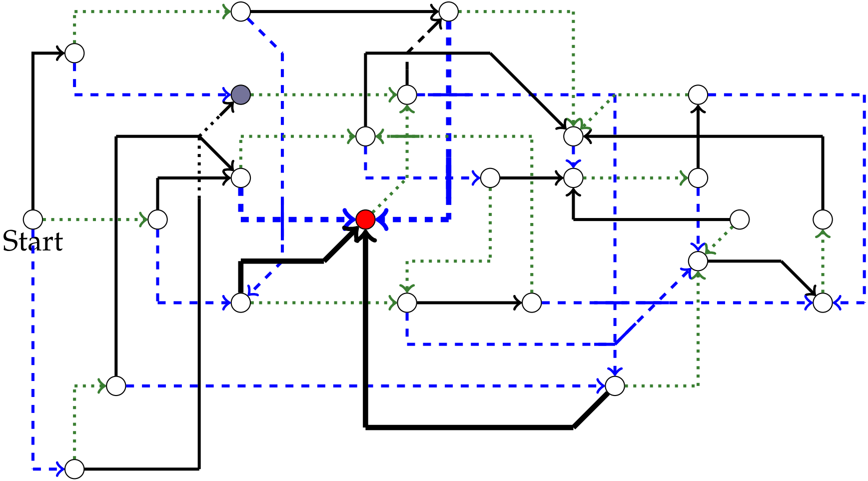
FINITE STATE AUTOMATA FOR COXETER GROUPS

Theorem (Brink-Howlett, 1993)

For each finitely generated Coxeter group W, there exists a finite state automaton which recognises the language of reduced words of W.

BRINK-HOWLETT AUTOMATON FOR $\tilde{B}_2$





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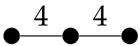

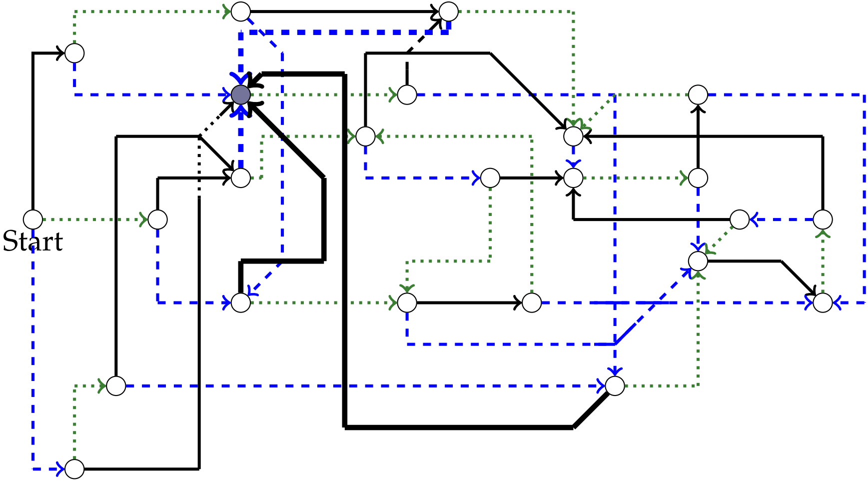

For which Coxeter systems is the Brink-Howlett automaton minimal?

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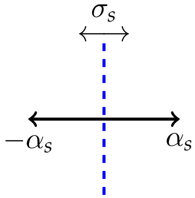

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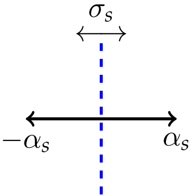

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- **Remark:** This is a faithful representation.

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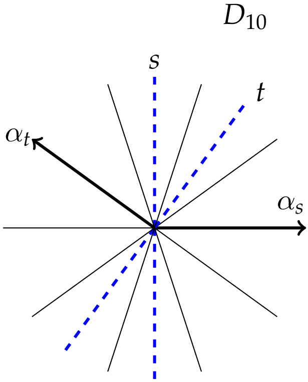

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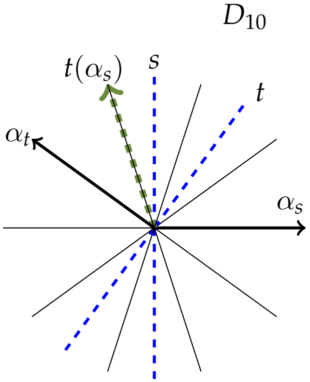

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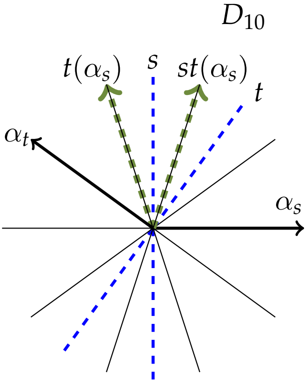

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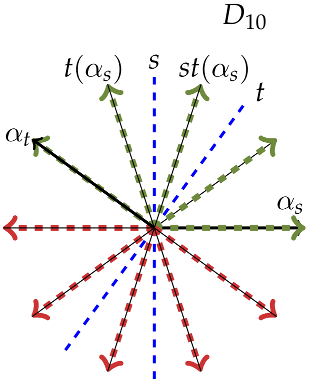

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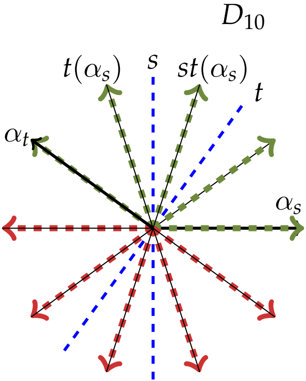

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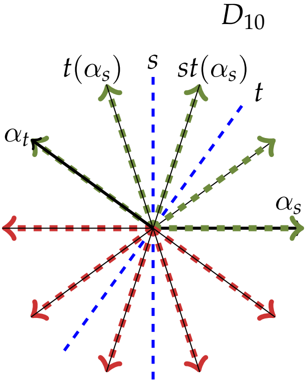
- Any root $\alpha \in \Phi$ is either a **positive** or **negative** linear combination of the basis of simple roots.



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- Where to direct the edge s from a state representing w to the state representing ws.

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Conjecture (Hohlweg-Nadeau-Williams, 2016)

The Brink-Howlett automaton $\mathcal{A}_{BH}$ is minimal if and only if

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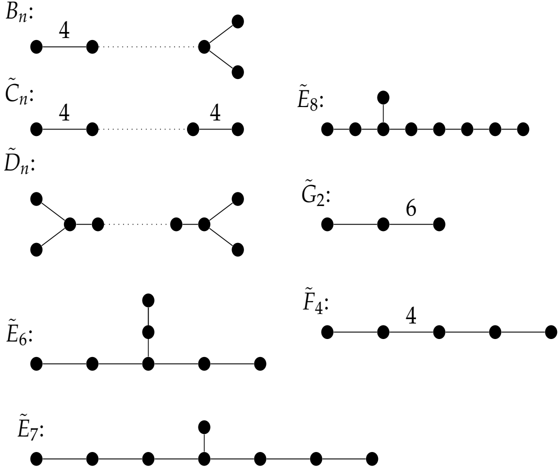
- For $\alpha \in \Phi^+$, can write $\alpha = \sum_{s \in S} c_s \alpha_s$ with $c_s \geq 0$.
- The *support* of $\alpha \in \Phi$ is the set $J(\alpha) = \{s \in S \mid c_s \neq 0\}$. Eg. if $\alpha = \alpha_s + \alpha_t$ then $J(\alpha) = \{s, t\}$.

Define $\mathscr{X}$ to be the following set of Coxeter graphs:

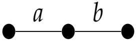
$$\mathscr{X} = \{\text{affine irreducible}\} \bigcup \{\text{compact hyperbolic}\}.$$

with no circuits or infinite bonds.

Affine irreducible graphs (other than $\tilde{A}_n$)



Compact hyperbolic graphs with no curcuits or infinite bonds



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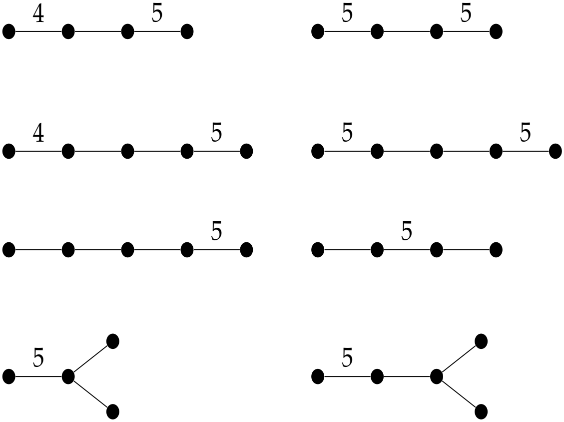


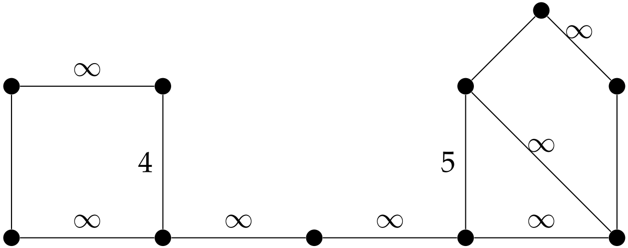
Theorem (J. Parkinson, Y.Y, 2018)

Let (W, S) be a finitely generated Coxeter system. The following are equivalent:

(1) *The Brink-Howlett automaton $\mathcal{A}_{BH}$ is minimal.*

(2) *The Coxeter graph of (W, S) does not have a subgraph contained in $\mathscr{X}$.*

(3) *The set of elementary roots is $\mathscr{E} = \Phi^+_{\mathrm{sph}}$.*



The automaton $\mathcal{A}_{BH}$ is minimal for this Coxeter group!

MINIMAL AUTOMATON

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- The automaton $\mathcal{A}_{BH}$ is minimal if and only if $T(w) = T(v)$ whenever $\mathscr{E}(w) = \mathscr{E}(v)$.

Outline of Proof

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Lemma

Let (W, S) be a finitely generated Coxeter system. If there exists $J \subset S$ and $t \in S$ such that:

(i) *J is spherical, and*

(ii) *$J \cup \{t\}$ is not spherical, and*

(iii) *$w_J(\alpha_t) \in \mathscr{E}$, where w_J is the unique longest element of W_J.*

Then $T(t \cdot w_J) = T(w_J)$ and $\mathscr{E}(w_J) \neq \mathscr{E}(t \cdot w_J)$.

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- Fact: Let φ be the highest root of Φ_0. There is a unique simple root α_t, such that $\langle \varphi, \alpha_t \rangle = 1$ and $\langle \varphi, \alpha_i \rangle = 0$ for all other simple roots α_i.

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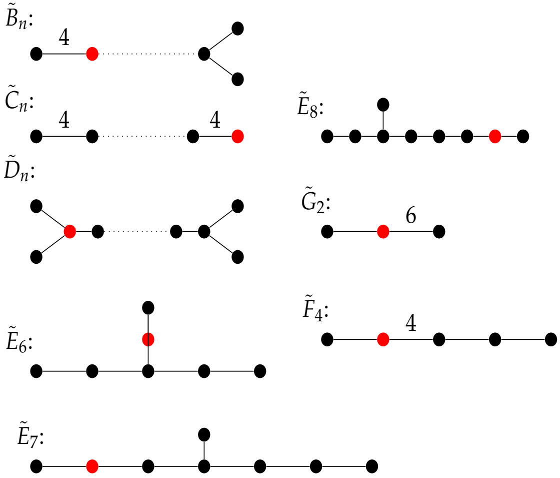
- Let $t \in S$ be the simple reflection associated to α_t.



Take t to be the red dot and $J = S \setminus \{t\}$. Then $T(tw_J) = T(w_J)$ and $\mathcal{E}(tw_J) \neq \mathcal{E}(w_J)$.

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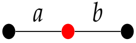

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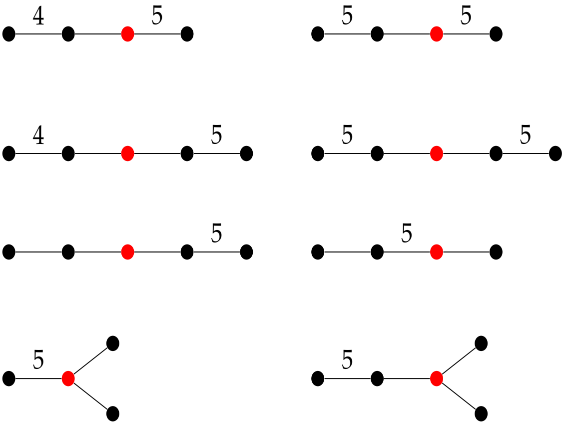

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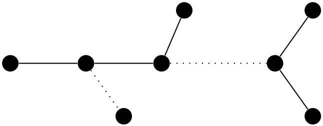
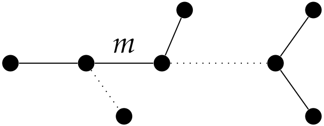
- Using a key result of Brink, $\Gamma(J(\alpha))$ must be a tree with no infinite bonds.



- Let e_m be an edge with maximal edge label m of $\Gamma(J(\alpha))$.



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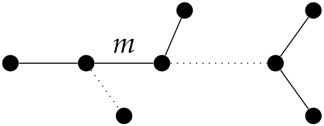
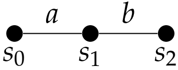


- Nonexistence of sub-graphs of type $\tilde{G}_2$ and



where $a, b < \infty$, $\frac{1}{a} + \frac{1}{b} < \frac{1}{2}$.

$\implies m < 6$.

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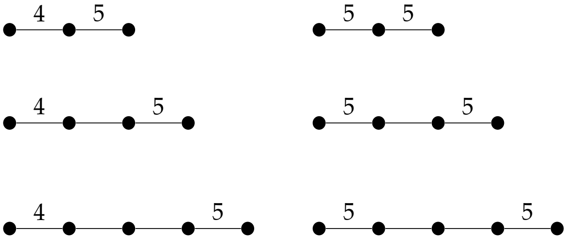
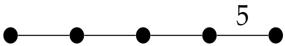


$\Longrightarrow \; d(e_m, e_j) > 3.$

- However, nonexistence of



gives a contradiction.

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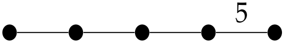
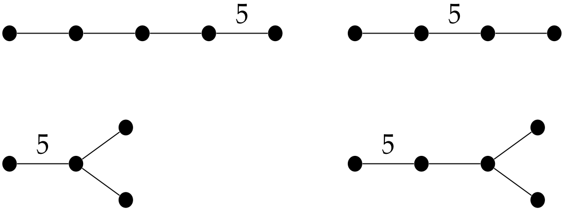


gives a contradiction.

- Therefore, there is a unique edge label of $m = 5$.

- But nonexistence of



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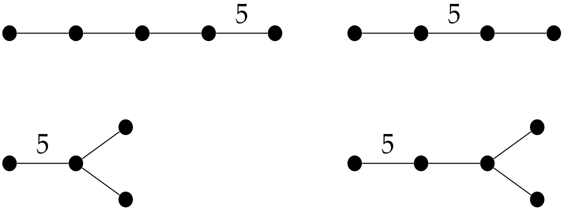

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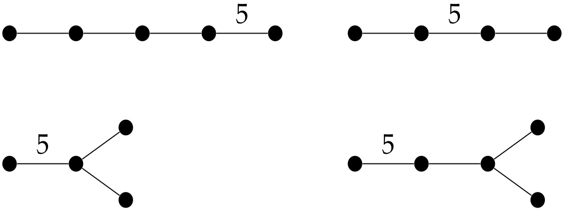

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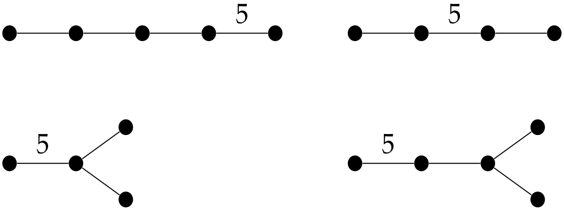

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- But these are finite Coxeter groups. Contradiction.
- Cases $m = 4$ and $m = 3$ are similar.

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- Proven by Hohlweg, Nadeau and Williams (2016).
- Using the key fact that $\mathcal{A}_{BH}$ is minimal for finite Coxeter groups.

CONNECTIONS WITH MINIMAL AUTOMATA

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- Hence good reasons to explore more of this story...

Thank you.