Infinite primitive permutation groups, cartesian decompositions, and topologically simple locally compact groups

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August 2020

Background

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Can do geometric group theory: two Cayley-Abels graphs for G are quasi-isometric; ends of groups

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Under their "natural" topology closed, transitive, subdegree-finite permutation groups are tldc	$\leftrightarrow$	Under their "natural" permutation representations tdlc groups are closed, transitive and subdegree-finite.

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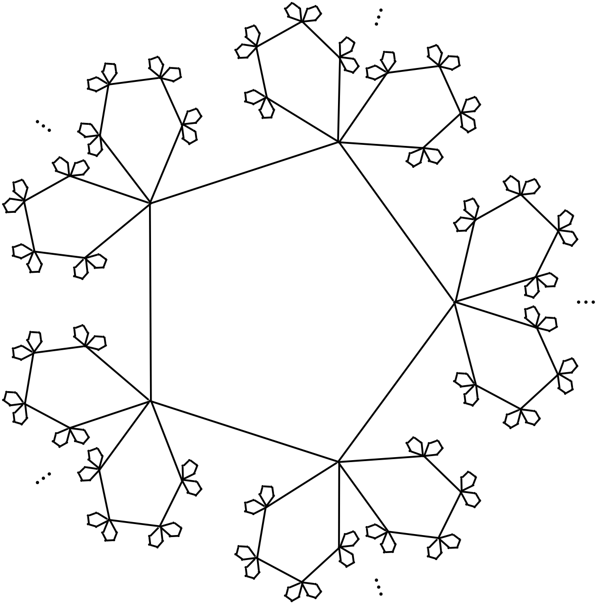

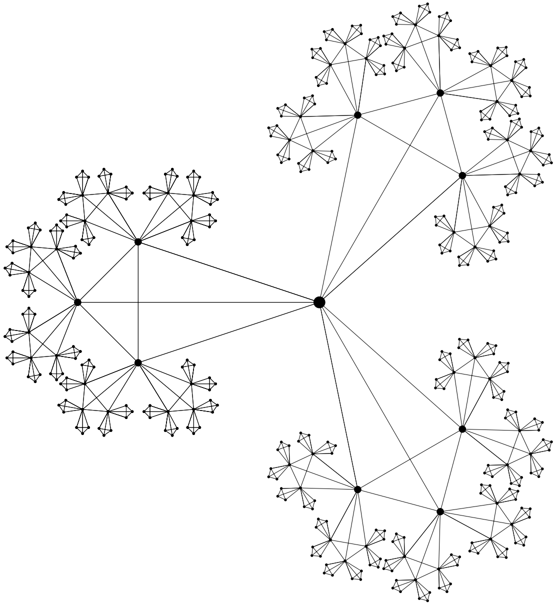

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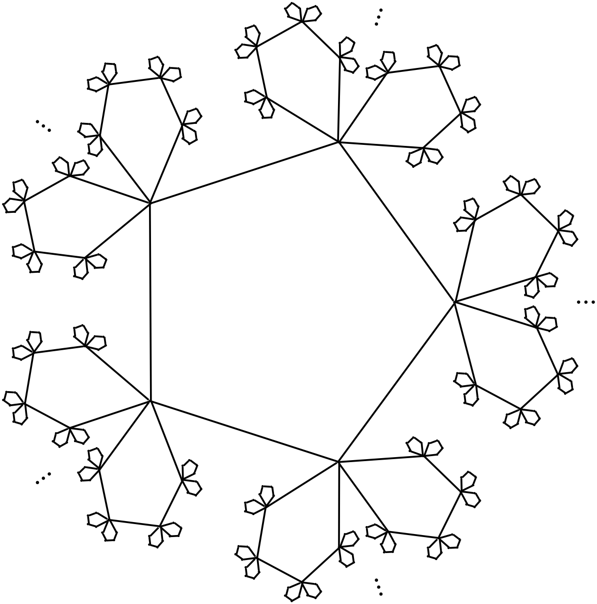

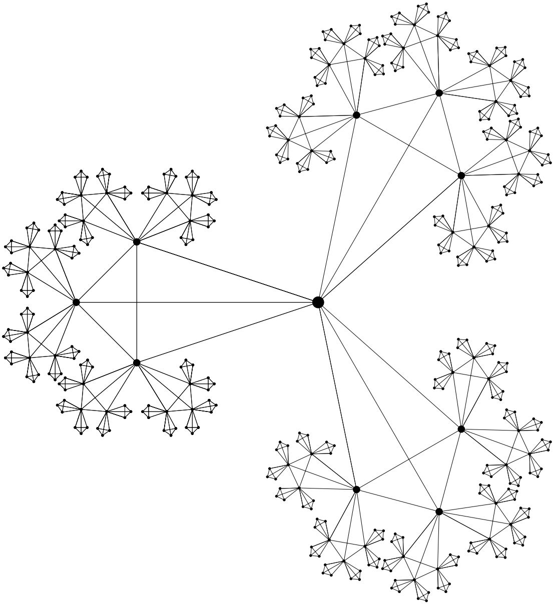

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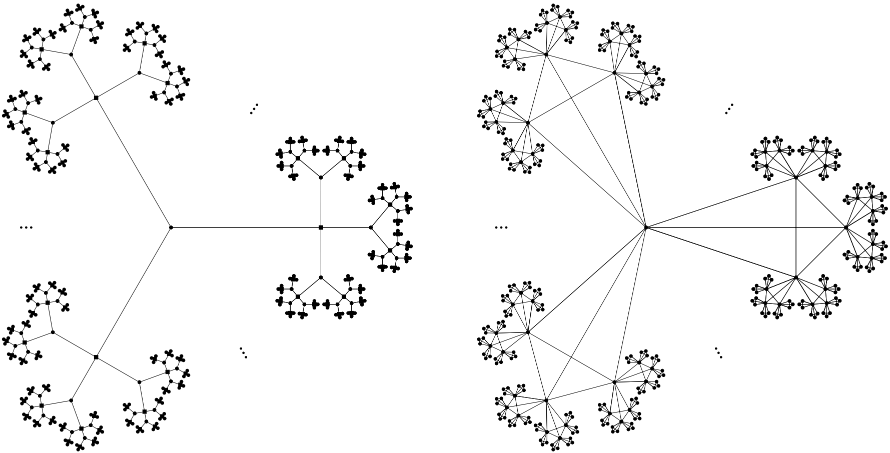
Note: Any subgroup $G \leq \mathrm{Aut}(\Gamma(m, \Lambda))$ has a faithful action on the $(|\Lambda|, m)$-biregular tree ...



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- Conversely, if $G \leq \mathrm{Sym}(X)$ preserves some $\mathcal{E} = \{\Sigma_1, \ldots, \Sigma_m\}$ on X, then G is a subgroup of $\mathrm{Sym}(\Sigma_1)\mathrm{Wr}S_m$.

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Let C be the cube $\{(a, b, c) : a, b, c \in \{1, 0\}\}$.

Partition C according to the x-coordinate:

$$\Sigma_1 := \{\{\text{elements whose } x \text{ coordinate is } 0\},\\ \{\text{elements whose } x \text{ coordinate is } 1\}\}$$

Now do the same for the y-coordinate (Σ_2) and z-coordinate (Σ_3)

The "points" in C can be recovered by taking intersections of parts:

$$|\gamma_1 \cap \gamma_2 \cap \gamma_3| = 1 \qquad \text{for all } \gamma_i \in \Sigma_i$$

- $\mathcal{E}$ is a nontrivial homogeneous cartesian decomposition of C
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- The S_3 interchanges the 3 components. Hence $\mathrm{Aut}(C)$ preserves $\mathcal{E}$.

Tdlc groups with maximal compact open subgroups & one-ended groups in $\mathscr{S}$

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We can follow the proof to obtain a structure theorem for $G\dots$

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  (here $\psi : N_{\hat{U}}(K) \to N_{\mathrm{Stab_{Sym(Y)}(y)}}(K)$ is a known homomorphism)

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- Nondiscrete examples: Certain completions of Kac-Moody groups (Caprace, Marquis, Rémy)

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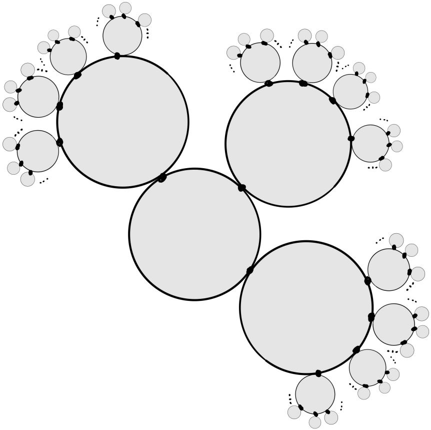
(This decomposition eventually halts after finitely many steps)

So $G/\!/U \leq_{\mathrm{prim}} (((H_0 \mathrm{Wr} F_1) \boxtimes F_2) \mathrm{Wr} F_3 \cdots \boxtimes F_{n-1}) \mathrm{Wr} F_n$

H_0 is OAS or finite primitive & non-reg
F_i are finite transitive



16 A test for one-ended groups in $\mathscr{S}$

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then the monolith of $G/\!/U$ is a one-ended group in $\mathscr{S}$.