Summer Projects 2019/2020

Symmetry Research Group - Zane Marsh
Colouring Platonic Solids: An Exercise In Madness

Problem Outline

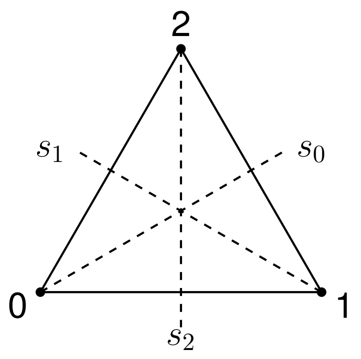
It is a truth universally acknowledged that mathematicians succumb to madness when left alone for extended periods of time. This poster is the result. Our poor mathematician (me) decided to find how many ways they could uniquely colour the platonic solids, using k colours.
The main issue in this was discovering how to factor in the various rotations, symmetries, and movements of these objects and how this affected the uniqueness of a colouring. Fortunately, there is an area of mathematics specifically dedicated to study of symmetries and groups of symmetries, fittingly called...

Group Theory

Defined broadly as the study of symmetry. Symmetries are the transformations that can be composed on a group. A group is a pair $(G, \circ)$ where G is a set and $\circ$ an operator maps from $G \times G \to G$, which satisfies the following axioms.

i. (Associative) For all $g_1, g_2, g_3 \in G$, $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$
ii. (Neutral Element) There exists $e \in G$ such that for all $g \in G, e \circ g = e = g \circ e$
iii. (Inverse Elements) For all $g \in G$, there exists $g' : g \circ g' = e$

A simple example of a group can be seen in the symmetries of an equilateral triangle. It has three rotational symmetries by 0, 120 and 240 degrees which we will label r_0, r_1 and r_2, and three reflectional symmetries about the vertices which we will call s_0, s_1 and s_2. These can be 'added' or composed together and form results inside the group. It should be noted that composition of these elements is not always commutative, $g_1 \circ g_2 \neq g_2 \circ g_1$.



$\circ$	r_0	r_1	r_2	s_0	s_1	s_2
r_0	r_0	r_1	r_2	s_0	s_1	s_2
r_1	r_1	r_2	r_0	s_2	s_0	s_1
r_2	r_2	r_0	r_2	s_1	s_2	s_0
s_0	s_0	s_1	s_2	r_0	r_1	r_2
s_1	s_1	s_2	s_0	r_2	r_0	r_1
s_2	s_2	s_0	s_1	r_1	r_2	r_0

$**$ Note for the addition table: Compose the element in the top row first, then apply the column symmetry.

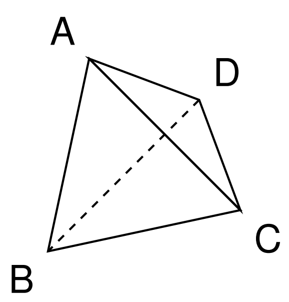
Burnside's Theorem

Now that we have established the basics of group theory, we can look at how colouring affects the platonic solids.
In essence we are applying Burnside's theorem to the group of symmetries of each platonic solid. The theorem states that:

$$|G \backslash K| = \frac{1}{|G|} \sum_{g \in G} |K^g|.$$

While looking complicated, in essence, this formula finds the number of ways you can uniquely colour an object by looking at how each g, or rotation, of a solid, would interact with $k \in K$, the group of colours, essentially looking at what combination of colours does the shape have to be in order to look identical under the group's symmetries. We then add up all of these and divide by the number of symmetries to compensate for over-counting.

The Tetrahedron



The Tetrahedron has 4 faces, each of which can be oriented 3 ways, giving a total of 12 symmetries. This means we have to examine how each of these rotations would affect the colouring of the sides.

- The Neutral Element: Does nothing, meaning each face is fixed to itself. This means each face can be represented k ways, giving k^4 options
- The 11 remaining elements: All fix one face and rotate about that face. One face is fixed giving k options, the remaining 3 faces are all placed in a cycle, where they all must be the same colour, contributing k again, multiplying these together we obtain $11k^2$

The full equation for number of ways to colour a tetrahedron is the sum of these elements divided by the number of symmetries:

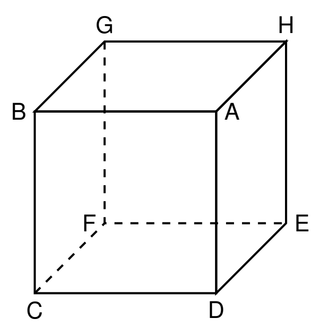
$$\frac{k^4 + 11k^2}{12}$$

A way of checking this formula is correct is to see how these rotations transform the vertices of the tetrahedron. These rotations can be denoted as follows:

(A)(B)(C)(D), (AD)(BC), (AB)(CD), (AC)(BD),
(A)(BCD), (A)(BCD), (B)(ACD), (B)(ADC),
(C)(ABD), (C)(ADB), (D)(ABC), (D)(ACB).

The Rotation (A)(BCD), means that A is fixed, while B-C-D cycles, so B goes to C, C goes to D and D goes to B. Due to the tetrahedron being self dual, each vertex is directly related to its opposite face.

The Cube



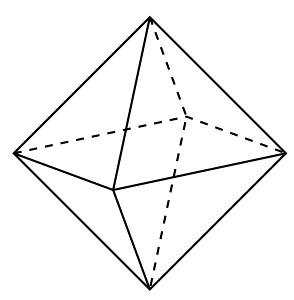
The Cube having 6 faces and 4 ways of rotating each face, has 24 symmetries. Now, how do these 24 rotations affect the colouring of the cube?

- The neutral element fixes all faces, giving k^6 options for colouration.
- 6 rotations by $\pm\pi/4$ around the centers of the three opposite faces. Fixes the opposite faces as themselves (k^2), the remaining 4 faces are fixed as one colour (k), contributing $6k^3$ in total.
- 3 rotations by $\pi/2$ around the center of opposite faces. This fixes the opposite faces being rotated about as themselves (k^2), the 4 remaining faces are fixed to their opposite face (k^2), giving $3k^4$.
- 8 rotations by $\pm\pi/3$ around the 4 pairs of opposite vertices. This fixes two cycles of three adjacent faces giving $8k^2$ options.
- Rotation by π about the center of the six pairs of opposite edges. This fixes one set of opposite faces to each other (k). The remaining 4 sides are fixed by having two sets of two adjacent faces fixed to each other (k^2), giving $6k^3$.

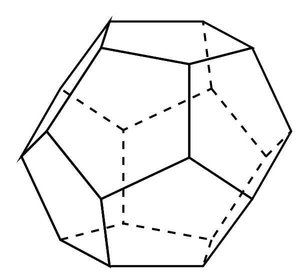
Adding these together and dividing by 24, we obtain:

$$\frac{k^6 + 3k^4 + 12k^3 + 8k^2}{24}$$

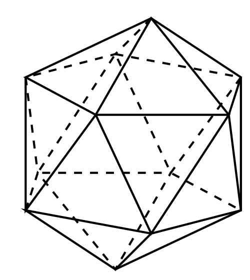
The Octohedron, Dodecahedron and Icosahedron



$$\frac{k^8 + 17k^4 + 6k^2}{24}$$



$$\frac{k^{12} + 15k^6 + 44k^4}{60}$$



$$\frac{k^{20} + 15k^{10} + 20k^8 + 24k^4}{60}$$

Thus concludes our foray into group theory. If you wanted an exercise to torment yourself with I would recommend attempting to inductively prove each expression is a whole number for all positive k, the tetrahedron is easiest, the rest are a headache. You could also try and formulate a composition table for the rotations of these solids (I would not recommend this), or you can construct a representation of them, i.e. a group of matrices, that acts exactly the same way under composition (far easier than you think).

https://zerodimensional.group