ICOSAHEDRON SYMMETRY GROUP AND ROTATION MATRICES

Representation theory
A representation of a group G is a homomorphism from G into the general linear group of a vector space V. \( G \rightarrow GL(V) \)
For example, mapping \( S_3 \) to \( GL(\mathbb{R}^3) \)
\( S_3 \) is the symmetry group of the set \{1, 2, 3\}, in other words, the permutations of the set \{1, 2, 3\}. We can see that the permutations \{(1,2), (2,3) and (1,3)\} generate the six elements of \( S_3 \).
In \( \mathbb{R}^3 \), there are 3 axis \( (x, y, z) \) which can represent number 1, 2 and 3.
Use matrices multiplication, we can switch 3 axis around as follows:
Firstly, I have identity matrix as \( [1 0 0] \) \( [0 1 0] \) \( [0 0 1] \)
Then, switch x and y by using \( [0 1 0] \) \( [1 0 0] \) \( [0 0 0] \)
Similarly, switch y and z by using \( [0 0 1] \) and switch x and z

Here, we combine group theory, Euler’s rotation theorem and representation theory to create basic rotation matrices

**Rotation by \( 2\pi/5 \) around axis \( (0,1,\varphi) \)**

\[
R_{2\pi/5} = \begin{bmatrix}
\cos \left(\frac{2\pi}{5}\right) & -\frac{\varphi}{\sqrt{1+\varphi^2}} \sin \left(\frac{2\pi}{5}\right) & \frac{1}{\sqrt{1+\varphi^2}} \sin \left(\frac{2\pi}{5}\right) \\
\frac{\varphi}{\sqrt{1+\varphi^2}} \sin \left(\frac{2\pi}{5}\right) & \cos \left(\frac{2\pi}{5}\right) & -\frac{\varphi}{\sqrt{1+\varphi^2}} \cos \left(\frac{2\pi}{5}\right) \\
\frac{1}{\sqrt{1+\varphi^2}} \sin \left(\frac{2\pi}{5}\right) & \frac{\varphi}{\sqrt{1+\varphi^2}} \cos \left(\frac{2\pi}{5}\right) & \cos \left(\frac{2\pi}{5}\right)
\end{bmatrix}
\]

**Rotation by \( 2\pi/3 \) around axis \( (1,1,1) \)**

\[
R_{2\pi/3} = \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 1
\end{bmatrix}
\]

**Rotation by \( \varphi \) around Z axis**

\[
R_{\varphi} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

The neutral element is \( 3 \times 3 \) identity matrix.

Elements generated from \( R_{2\varphi/5} \), \( R_{2\varphi/3} \), \( R_{2\varphi} \), \( R_{2\varphi}^{-1} \). Elements generated from \( R_{2\varphi/5} \), \( R_{2\varphi/3} \), \( R_{2\varphi} \).

Connection between group theory and quantum mechanics

Representation theory is fundamental to quantum mechanics. It allows physicists to gain information about quantum mechanical state spaces when a group acts on a system. For example, quantum particles such as hydrogen atom can be represented by a complex-valued “wavefunction” – Lie Group. It acts on such wavefunctions by pointwise phase transformations of the value of the function. This allows us to understand and describe how particles interact with electromagnetic fields. Otherwise, it would be very hard for scientists to grasp the unworldly strange behaviour of the quantum particles.