0-Dimensional Symmetry

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May 15th 2018

Topology, Algebra, Geometry and Symmetry

A random walk

Totally disconnected locally compact groups

Topology and Algebra

0-dimensional symmetry

totally disconnected locally compact groups

Derivative

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Integral

$$\lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*) \Delta x_i$$

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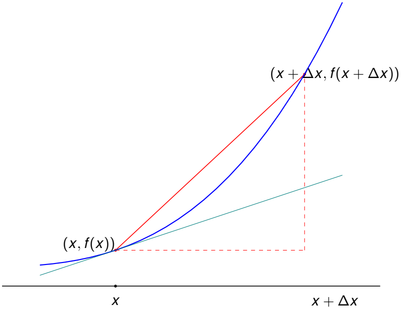
Topology and Algebra and Geometry

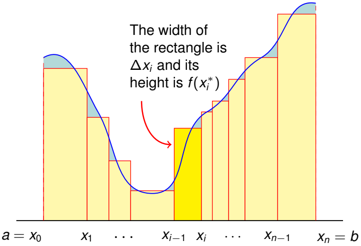
Derivative

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Integral

$$\lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$





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Area is invariant under translation

Length is invariant under translation

Topology, measure and symmetry

Shared features of the line, plane and 3-dimensional space.

Measure: length, area and volume (additive).
Topology: distance $d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$.
Symmetry: groups of translations.

F. Klein (1872): *geometric properties are characterised by their remaining invariant under a group of transformations*.

Locally compact group

symmetry + measure $\overset{\text{A. Weil}}{\Longrightarrow}$ symmetry + topology

symmetry + measure $\Longleftarrow$ symmetry + topology (J. von Neumann, A. Haar)

Locally compact groups have all the features needed for calculus and for harmonic analysis.

Lie

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Examples of locally compact groups

SO(3) - rotations of the sphere

$\mathbb{T}^2$, $U(5)$, $GL(2,\mathbb{R})$, $SL(3,\mathbb{C})$, $Sp(6,\mathbb{R})$, $O(2,3)$, $\mathrm{Spin}(3)$, $\mathbb{H}$, $\mathbb{R}^{1,3} \rtimes O(1,3)$, ... and many others.

Groups are abstract objects which may be distinguished, or seen to be the same, by their algebraic properties and named.

connected

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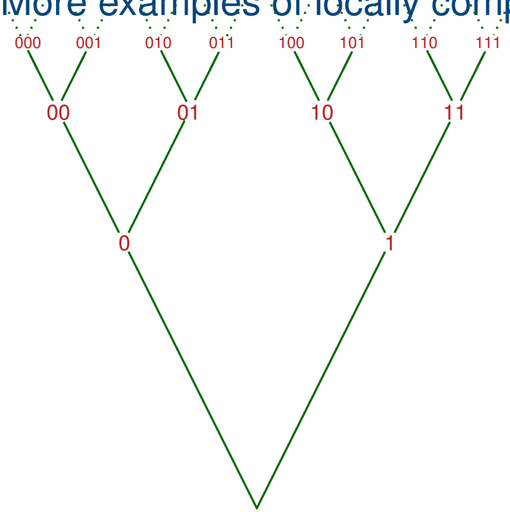
More examples of locally compact groups



Symmetries of the binary rooted tree

$\mathbb{Z}_2$, $\mathbb{Q}_5$, $\mathbb{Q}_{13}^\times$, $\mathbb{F}_7[[t]]$, $\mathbb{F}_{16}((t))$, $O(3, \mathbb{Q}_5[\sqrt{2}])$, $Sp(6, \mathbb{F}_{27}((t)))$, $\mathbb{H}(\mathbb{Q}_3)$, $SL(3, \mathbb{Z}_3)$, $PGL(5, \mathbb{Z}_7[\sqrt[4]{-1}])$, Kac-Moody $\widehat{G(A)}$, $Aut(\Gamma)$, $AAut(\Gamma)$, ... and still others.

Each group determines a geometry which has that group of symmetries.

Random walks

Random walk on G: Step from one point in G to another random point, according to a fixed probability distribution. rw

Random walks model diffusion. Smoke in a closed room diffuses to a uniform distribution. Smoke outside disperses.

Conjecture: The same occurs for random walks on groups.

K. H. Hofmann, A. Mukherjea, 'Concentration functions and a class of noncompact groups', *Math. Ann.* **256** (1981), 535–548.

Reduced the conjecture to one about the structure of G and proved it for connected G.

Connected locally compact groups

Connected: between any two points there is a continuous path (a topological property).

Hofmann & Mukherjea used *approximation by Lie groups*.

locally compact, connected group

Hilbert's Fifth Problem
$\Longrightarrow$
D. Montgomery & L. Zippin
A. Gleason, H. Yamabe

Lie group, functions have a derivative

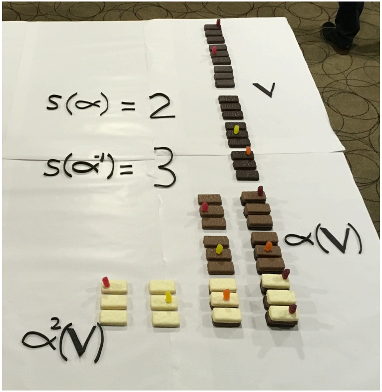
Lie groups, and connected groups, are essentially matrix groups. Eigenvalues and eigenvectors are used to analyse their structure.

Totally disconnected locally compact groups

Totally disconnected: between no two distinct points is there a continuous path.

Steps to proof of Hofmann-Mukherjea conjecture.

read paper	seek counter-example
+ 2 weeks	fail to find one
+ 2 months	see iteration stabilise
+ 1 week	product picture emerges, basic lemma
+ 2 months	picture completed, conjecture proved



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The scale function on a t.d.l.c. group

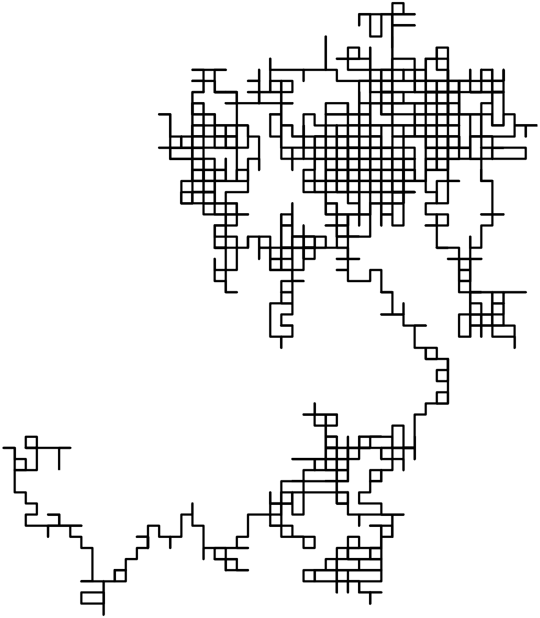
The proof showed that there is a positive integer-valued function on G called the *scale* which encodes structure of the group. The scale plays the role of eigenvalues in the proof.

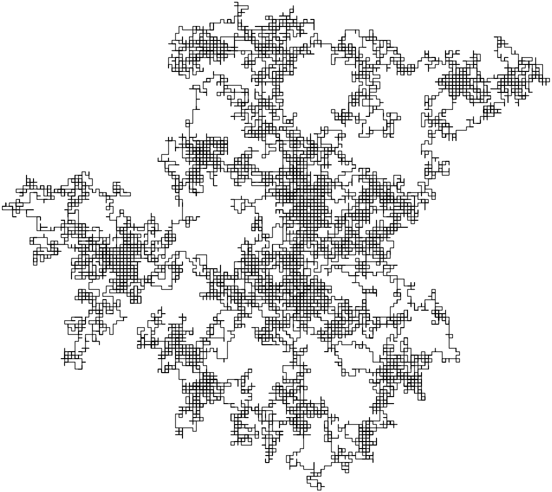
- Scale methods have been used to solve other problems in ergodic theory and concerning arithmetic groups.
- The analogy with eigenvalues has been borne out by additional properties of the scale shown since.
- The scale continues to inspire advances in the understanding of t.d.l.c. groups and many questions remain to be answered.

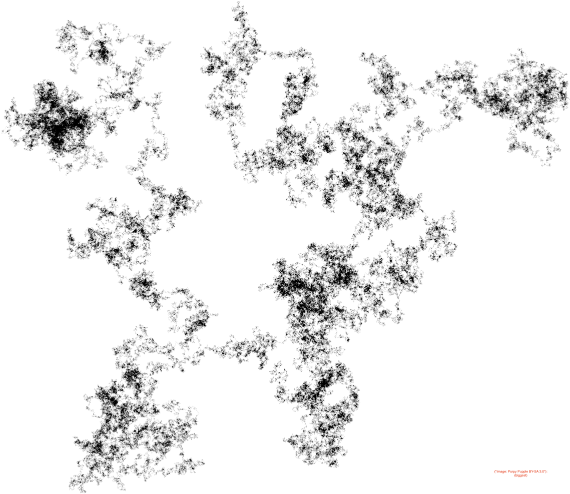
Other recent advances on t.d.l.c. groups

- A theory of local structure of t.d.l.c. groups has been developed by P.-E. Caprace, C. Reid and W. which classifies the groups into five types.
- Significant progress with methods for decomposing t.d.l.c. groups into smaller pieces has been made by C. Reid and P. Wesolek which builds on earlier work of P.-E. Caprace and N. Monod. The corresponding methods for Lie groups use dimension as a measure of the size of the group, but that is not available for 0-dimensional groups.
- Much remains to be done to combine these different approaches into a more complete picture of general t.d.l.c. groups.

Thank you for your attention







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