Cocycles on Trees
and
Translation-Like Action on LC-groups

Thibaut Dumont

University of Jyväskylä

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University of Newcastle, Australia

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Part I: Cocycles on trees

Ph.D. Thesis and on going work

Motivations: Cohomology of Groups

Let (V, π) be a linear representation of a group G:

$$g \mapsto \pi(g) \quad \text{homomorphism into } \mathrm{GL}(V)$$

- The **cohomology groups** are important invariants of G:

$$\mathrm{H}^*(G, V), \qquad * = 0, 1, 2, \ldots$$

Cohomolgy may computed with the standard **cochain** complex $\mathrm{C}^*(G, V)$ of all maps

$$f : G^n = G \times \cdots \times G \to V,$$

together with the operator $d : \mathrm{C}^n(G, V) \to \mathrm{C}^{n+1}(G, V)$:

$$(df)(g_0, \ldots, g_n) = \sum_i (-1)^i f(g_0, \ldots, \widehat{g_i}, \ldots, g_n).$$

Naturally, $d^2 = 0$.

Motivations: Cohomology of Groups

We use the **homogeneous** complex:

Definition (Cocycles and coboundaries)

- n-**cochain:** $f : G^{n+1} \to V$ with

$$f(gg_0, \dots, gg_n) = \pi(g) f(g_0, \dots, g_n) \qquad \text{(homogeneity)}.$$

- n-**cocycle** is an n-cochain $f : G^{n+1} \to V$ satisfying:
  - $df = 0$,
  - $f(gg_0, \dots, gg_n) = \pi(g) f(g_0, \dots, g_n)$.
- n-**coboundary** is $f = dc$ where c is an $(n-1)$-cochain.

$$\mathrm{H}^n(G, V) = \frac{\{n - \text{cocycles}\}}{\{n - \text{coboundaries}\}}.$$

Motivations: Cohomology of Groups

Let (V, π) be a unitary representation of a locally compact group G:

$$\langle \pi(g)v, \pi(g)v' \rangle = \langle v, v' \rangle.$$

We may ask cochains to be **continuous**:

- $\mathrm{H}^1_{\mathrm{c}}(G, V)$ classifies the affine isometric actions of G on V with linear part π up to conjugation by a translation.

Or impose a growth condition on the function

$$(g_0, \ldots, g_n) \mapsto \|f(g_0, \ldots, g_n)\|_V$$

to obtain other cohomology theories:

- A uniform bound defines **bounded cohomology** $\mathrm{H}^*_{\mathrm{b}}(G, V)$ (Gromov, Burger–Monod).
- A polynomial bound with respect to distances $d(g_i, g_j)$ defines **polynomially bounded cohomology** $\mathrm{PH}^*(G, V)$ for G compactly generated. (Connes–Moscovici, Ogle).

(Polynomially) Bounded Cohomology

Bounded cohomology:

- (Johnson) **Amenability** of G is characterized by vanishing of $\mathrm{H}_{\mathrm{b}}^{*}(G, V)$ for a suitable family of coefficients V.
- (Gersten, Mineyev) **Gromov hyperbolicity** of G is characterized by $\mathrm{H}_{\mathrm{b}}^{2}(G, V) \to \mathrm{H}^{2}(G, V)$ being injective for a suitable family of coefficients V.
- (Brooks) $\dim_{\mathbf{R}} \mathrm{H}_{\mathrm{b}}^{2}(\mathbf{F}_2, V) = \infty$.
- Generally hard to compute. (No example of a countable group where $\mathrm{H}_{\mathrm{b}}^{*}(G, \mathbf{R})$ is known and non trivial.)

Polynomially bounded cohomology:

- (Connes–Moscovici) Novikov conjecture for hyperbolic groups.
- Ogle–Ji–Ramsey extended notion of $\mathcal{B}$-**bounded cohomology**.
- Hard to compute and few examples.

Goal

Let G be an almost-simple p-adic algebraic group, say $\mathrm{SL}_{n+1}(\mathbf{Q}_p)$, and $V = \mathbf{St}$ its **Steinberg representation**.

- $V = \mathbf{St}$ is irreducible, unitary, and $\mathrm{H}^n(G, \mathbf{St}) \neq 0$.
- In fact, Casselman showed $\mathrm{H}^n(G, \mathbf{St}) = \mathbf{C}$.
- In 2003, Klingler built a natural **volume cocycle** vol_G whose class generates $\mathrm{H}^n(G, \mathbf{St}) = \mathbf{C}$.

Goal: Is **Klingler volume cocycle** vol_G polynomially bounded?

Problem (Monod, 2006)

'Quasify Klingler volume cocycle in order to obtain new cohomology classes with polynomial bounds in an appropriate coefficient module.'

Outline

Question: Is vol_G polynomially bounded with respect to a suitable distance on G?

- vol_G is constructed geometrically in the Euclidean building X associated to G, called the **Bruhat-Tits** building of G. In fact vol_G is derived from a volume cocycle vol_X defined on X.
- When $G = \mathrm{SL}_2(\mathbf{Q}_p)$, the building X is the $(p+1)$-regular tree T_{p+1} and $\mathrm{vol}_X = B$ is the **Busemann cocycle**:

$$B(x, y)(\xi) = \lim_{z \to \xi} d(y, z) - d(x, z)$$

The **volume cocycle** of G is then

$$\mathrm{vol}_G(g_0, g_1) = B(g_0 x, g_1 x)$$

for some origin $x \in X$.

Outline

General strategy:

1. $\mathrm{vol}_X : X \times \cdots \times X \to \mathbf{St}$ exists for (many) Euclidean buildings X.
2. vol_G is obtained from vol_X by translating the arguments of vol_X.
3. G is very close to X from a metric point of view.

"We may forget about G and work with X."

The geometry of the Euclidean buildings X is rich and hopefully sufficient to compute the norm of vol_X.

Blackboard: The space $V = \mathbf{St}$ and the norm $\| \cdot \|_{\mathbf{St}}$ are particularly delicate to compute in general: it uses a **Poisson type transform** introduced by [Klingler 2004] . But the geometric nature of vol_X gives the intuition of a polynomial bound.

Results

Let $X = T_{q+1}$ be a $(q+1)$-regular thick tree with the graph metric d and visual boundary ∂X. Let $B : X \times X \to \mathbf{St}$ denote the Busemann cocycle.

Theorem (D. 2016)

There are constants $L, K > 0$ satisfying:

$$4d(x_0, x_1) \leq \|B(x_0, x_1)\|^2_{\mathbf{St}} \leq L \cdot d(x_0, x_1) + K$$

Independently and in a more general setting:

Theorem (Gournay–Jolissaint, 2015)

There are constants $A, B > 0$ satisfying:

$$\|B(x_0, x_1)\|^2_{\mathbf{St}} = A \cdot d(x_0, x_1) + B \cdot (q^{-d(x_0,x_1)} - 1),$$

The method of Gournay-Jolissaint uses a discrete Laplacian and harmonic analysis on regular trees.

Results

Let $X = T_{q_0+1,q_1+1}$ be a semi-homogeneous tree and $B : X \times X \to \mathbf{St}$ the Busemann cocycle.

Theorem (Gournay–Jolissaint, D. 2018)

There are constants $L, K > 0$ satisfying:

$$\|B(x_0, x_1)\|_{\mathbf{St}}^2 \leq L \cdot d(x_0, x_1) + K$$

This gives a higher rank result for product of semi-homogeneous trees.

Corollary (D. 2018)

Let X be a direct product of $n \geq 2$ semi-homogeneous trees and let vol_X denote the volume cocycle of Klingler. There is a polynomial P of degree n satisfying:

$$\| \mathrm{vol}_X(x_0, \ldots, x_n)\|_{\mathbf{St}}^2 \leq P(d(x_i, x_j)),$$

for all $x_0, \ldots, x_n \in X$.

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Part II: Translation-Like Actions on LC-groups

Joint work in progress with Thibault Pillon, KU Leuven.

(Discrete) Groups

Problem (Burnside)

Does an infinite group necessarily contain $\mathbf{Z}$ *as a subgroup?*

No. [Golod-Shafarevich 1964]

Problem (vN-Day)

Does a non-amenable group necessarily contain F_2 *as a subgroup?*

No. [O'shanskii 1980] Tarski monsters: infinite torsion 2-generated non-amenable.

However, there are relaxed solutions using **translation-like actions**:

- Geometric vN-Day: [Whyte 1999]
- Geometric Burnside: [Seward 2014]

(Discrete) Tranlsation-Like Actions

Definition (Wobbling or Piecewise Translation)

A self bijection φ of a group G is **piecewise translation** or **wobbling** if there is a finite subset T such that $\varphi(x) \in xT$ for all $x \in G$.

The group of wobbling bijections is denoted $\mathscr{W}(G)$.

Definition (TL-action)

A **translation-like action** of a group Γ (e.g. $\mathbf{Z}$ or F_2) on a group G is a homomorphism $\Gamma \to \mathscr{W}(G)$ such that Γ acts freely on G:

$$\forall w \in \Gamma, \quad (\varphi_w(x) = x \implies w = e).$$

(No fixed point for any $w \neq e$.)

(Discrete) Groups

Let G be a finitely generated group with a left-invariant word metric d given by a Cayley graph X.

- Each left translation is an isometry for $(X, d) = (G, d)$.
- Each right translation is at bounded distance from the identity:

$$d(x, xg) = \ell(x^{-1}xg) = \ell(g), \qquad \text{uniformly bounded in } x.$$

$\mathscr{W}(G)$ is exactly the group of bijection at bounded distance from the identity.

Theorem (Whyte, 1999)

A finitely generated group is non-amenable if and only if it admits a translation-like action of the free group F_2.

Theorem (Seward, 2014)

A finitely generated group is infinite if and only if it admits a translation-like action of $\mathbf{Z}$.

LC-Groups

The next definition is due to F.M. Schneider [2017].

Definition (Clopen Piecewise Translation)

Let G be a locally compact group. A self bijection φ of G is **clopen piecewise translation** if there exists a finite subset T such that:

- $\varphi(x) \in xT$ for all $x \in G$,
- $P_t = \{x \mid \varphi(x) = xt\}$ is clopen for all $t \in T$.

The group of clopen piecewise translation bijections is denoted $\mathscr{C}(G)$.

Definition (Clopen TL-action)

A **clopen translation-like action** of a group Γ (e.g. $\mathbf{Z}$ or F_2) on a group G is a homomorphism $\Gamma \to \mathscr{C}(G)$ such that Γ acts freely on G with a **measurable** (strict) fundamental domain.

LC-Groups

Morally:

- Translation-like actions generalize the existence of certain subgroups: e.g. $\mathbf{Z}$, F_2, etc.
- Clopen translation-like actions generalizes the existence of certain **discrete subgroups**.

In a **connected** LC-group, a clopen piecewise translation is just a right translation.

LC-Groups

Theorem (Schneider, 2017)

A locally compact group is (topologically) non-amenable if and only if it admits a clopen translation-like action of F_2.

Theorem (D.-Pillon, 2018)

A compactly generated, locally compact group is non-compact if and only if it admits a clopen translation-like action of $\mathbf{Z}$.

Both rely on the connected case:

Theorem (Rickert, 1967)

A(n almost)-connected LC-group is (topologically) non-amenable if and only if it has a discrete subgroup isomorphic to F_2.

Theorem (?)

A(n almost)-connected CGLC-group is non-compact if and only if it has a discrete subgroup isomorphic to $\mathbf{Z}$.

[Gaillard/Karai math**overflow**]

Proof:

Let G be a CGLC-group. Since any $\varphi \in \mathscr{C}(G)$ preserves a right Haar measure μ:

$$\mathbf{Z} \curvearrowright G \text{ clopen TL-action } \implies \mu(G) = \infty \iff G \text{ non-compact.}$$

The converse is the interesting part. Suppose G non-compact. If we get a discrete $\mathbf{Z}$, we may reduce the structure of G.

- We may assume G is unimodular, otherwise it has a discrete $\mathbf{Z}$.
- The connected case implies G has a discrete $\mathbf{Z}$ or a compact open subgroup. Thus assume G has a Cayley-Abels graph X.
- If X has finitely many ends (1 or 2), Seward's theorem implies that $\mathbf{Z} \curvearrowright X$ translation-like. We can lift the action thanks to unimodularity.
- If X has infinitely many ends, by Stalling's Theorem for LC-group [Abels, Cornulier], G has a discrete F_2, hence a discrete $\mathbf{Z}$.

An Obstruction: Local Ellipticity

Given a clopen piecewise translation $\varphi \in \mathscr{C}(G)$ on a LC-group G.

- The orbits of φ are contained in the left cosets of a finitely generated subgroup $\langle T \rangle$.

Definition (Platonov 1966)

An LC-group G is **locally elliptic** if every compact subset is contained in a compact subgroup of G.

For G discrete, we say **locally finite**: every finitely generated subgroup is finite. There exist infinite locally finite groups (Hall's universal group).

- For CGLC-group, locally ellipticity $\iff$ compact.
- For σ-compact LC-group, G locally elliptic $\iff$ $\operatorname{asdim}(G, d) = 0$, for some adapted pseudo-metric, [Cornulier–de la Harpe].
- As Schneider observes, for a (discrete) group G: $\exists \mathbf{Z} \curvearrowright G$ TL $\iff$ G not locally finite $\iff$ $\operatorname{asdim}(G) > 0$.
- For general LC-groups, we don't know at the moment.

An Obstruction: Local Ellipticity

We are left with the questions:

- Are there LC-groups that admits no clopen translation-like actions of $\mathbf{Z}$? (Necessarily non-discrete.)
- Locally elliptic?
- TDLC?

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Thank you!