Automorphisms of forests of quasi-label-regular rooted trees

In preparation for *almost* automorphisms

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UoN

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We use the convention that automorphisms must preserve vertex labels.

Recall from the last talk

Let A be an $n \times n$ adjacency matrix with m non-zero entries. Then there are m types of quasi-label-regular rooted trees that obey this adjacency matrix except at the root which is missing a single neighbour.

We can create a $m \times m$ matrix M which describes what rooted trees remain after removing the root of each type of rooted tree.

The directed graph associated with this matrix is $X(M)$.

Recall from long ago?

The automorphism group of a rooted tree, T, where each vertex has the same number of children, say d, is an iterated wreath product.

$\mathrm{Aut}(T) \cong \mathrm{Aut}(T) \wr S_d$, where S_d is the symmetry group on the d children,

and so $\mathrm{Aut}(T) \cong \ldots \wr S_d \wr S_d = \wr_{i=1}^{\infty} S_d$.

Also, $\mathrm{Aut}(T)$ is transitive on the boundary of T, Ω_T.

In this talk

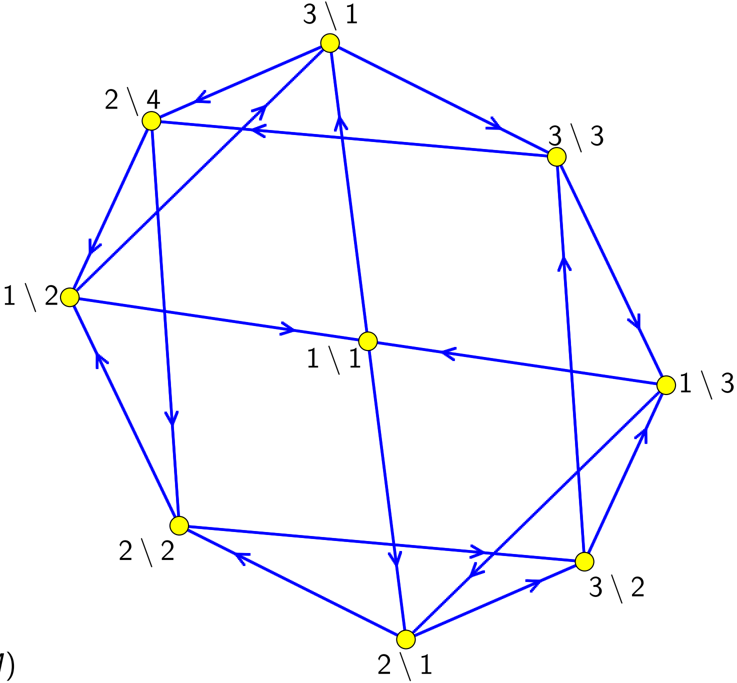
Let $i \setminus j$ be a rooted tree with adjacency matrix A. We investigate:

- When is $\mathrm{Aut}(i \setminus j)$ trivial?
- When is $\mathrm{Aut}(i \setminus j)$ an infinitely iterated wreath product of finite groups?
- When $\mathrm{Aut}(i \setminus j)$ isn't an infinitely iterated wreath product of finite groups, how many other Aut groups of rooted trees do we need in order to describe $\mathrm{Aut}(i \setminus j)$?

Trivial automorphism group

If the children of every vertex have a different label then nothing can be permuted!
$\mathrm{Aut}(i \setminus j) = \{1\}$ for all $i \setminus j \in P_A$.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, M = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$



$X(M)$

A rooted tree has trivial automorphism group when A is binary (0s and 1s only) or A only contains 0s, 1s, and 2s, but whenever there is a 2 in a row it is the only non-zero entry in that row.

e.g. $A = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$

Non-trivial automorphism group

If there are i, j, k such that $a_{ij} = 2$ and $a_{ik} > 0$ for $k \neq j$, or, there is an entry say $a_{ij} \geq 3$ then the automorphism group of each rooted tree present in A is non-trivial.

Note: this is still true when A is not well mixed since $i \setminus k$ and $j \setminus i$ must be in the sink.

Transitive?

$\mathrm{Aut}(i \setminus j)$ is not necessarily transitive on Ω_T.

Since we have to fix the root, an automorphism of $i \setminus j$ can only map $e_1 \in \Omega_T$ to $e_2 \in \Omega_T$ if there are rays $r_1 \in [e_1], r_2 \in [e_2]$ that both start at the root and have the same labels as each other along the entire ray. $r_1 \cap r_2$ could be just the root, a finite path or infinite (if $e_1 = e_2$).

Note, the condition that the rays start at the root is necessary.

If we followed these rays on $X(M)$ they would start at the root and have the same trajectory forever (if we count multiple edges as single weighted edges).

One level at a time

We can describe the automorphism group of a rooted tree of type $i \setminus j$ in terms of the children of the root and the rooted trees identified with those children:

$$\mathrm{Aut}(i \backslash j) = \begin{cases} \mathrm{Aut}(j \setminus i) \wr S_{a_{ij}-1} \times \prod_{a_{ik}>0, k \neq j} \mathrm{Aut}(k \setminus i) \wr S_{a_{ik}} & \text{if } a_{ij} \geq 2 \\ \prod_{a_{ik}>0, k \neq j} \mathrm{Aut}(k \setminus i) \wr S_{a_{ik}} & \text{if } a_{ij} = 1. \end{cases}$$

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- WM For well-mixed A: The automorphism group of each rooted tree is built out of the m automorphism groups.
- NWM For not well-mixed A: The automorphism group of each rooted tree of source type is built out of the m automorphism groups. The automorphism group of each rooted tree of sink type is built out of the $\frac{m}{2}$ automorphism groups of the sink type trees.

Good wreath?

If $i \setminus j$ has a finite fundamental domain/block/region (that preserves 'root-end' orientation), then we can describe $\mathrm{Aut}(i \setminus j)$ as an iterated wreath product of a finite permutation group. Do all quasi-label-regular rooted trees have a finite fundamental domain?

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For a given A, are there some rooted trees where this is possible and others where it isn't?

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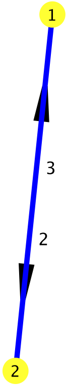
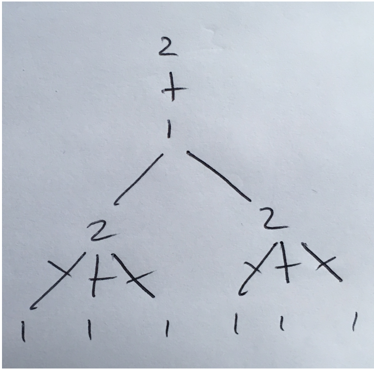
For a given A, are there some rooted trees where this is possible and others where it isn't?

For a given A, is there some minimal subset of trees $B \subset P_A$, such that the automorphism group of every rooted tree in P_A can be written in terms of the automorphism groups of the trees in B?

All with finite fundamental domain

$A = \left(\begin{smallmatrix} 0 & 3 \\ 4 & 0 \end{smallmatrix}\right)$





All cycles in $X(M)$ go through all vertices.

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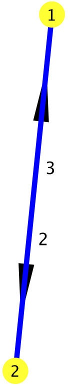
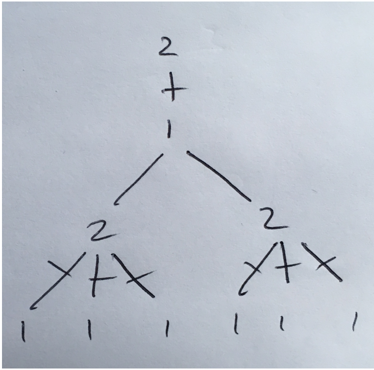
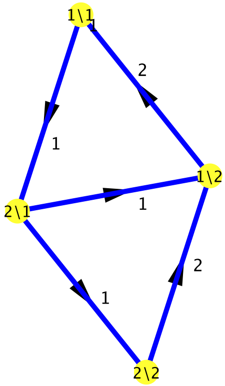


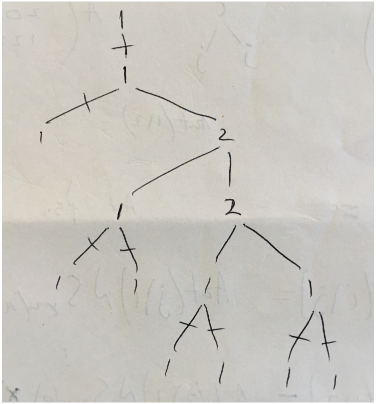


All cycles in $X(M)$ go through all vertices.
$\mathrm{Aut}(1 \setminus 2) \cong \mathrm{Aut}(1 \setminus 2) \wr S_3 \wr S_2 \cong \ldots \wr S_3 \wr S_2 \wr S_3 \wr S_2$
$\mathrm{Aut}(2 \setminus 1) \cong \mathrm{Aut}(2 \setminus 1) \wr S_2 \wr S_3 \cong \ldots \wr S_2 \wr S_3 \wr S_2 \wr S_3$

Finite fundamental domain

$A = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$





Note there is a loop on $1 \setminus 1$. There are no $1 \setminus 1$-avoiding cycles (but there are cycles that avoid $1 \setminus 2, 2 \setminus 1$ or $2 \setminus 2$).

Finite fundamental domain

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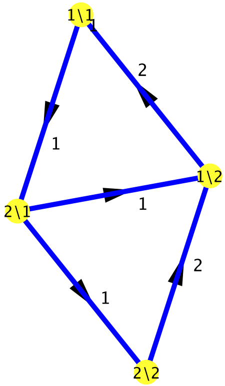
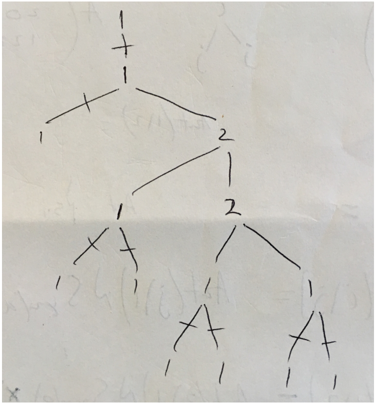




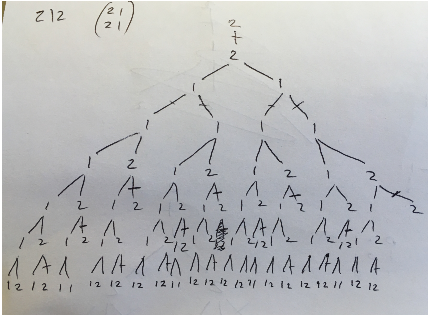
Note there is a loop on $1 \setminus 1$. There are no $1 \setminus 1$-avoiding cycles (but there are cycles that avoid $1 \setminus 2, 2 \setminus 1$ or $2 \setminus 2$).

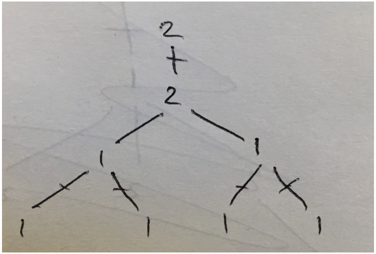
$\mathrm{Aut}(\partial FFD) \cong \{1\} \times S_2 \times (S_2 \wr S_2)$,

$\mathrm{Aut}(1 \setminus 1) \cong \ldots \wr_{\partial FFD} \mathrm{Aut}(\partial FFD) \cong (\ldots \wr_{\partial FFD(1 \setminus 1)} \{1\} \times S_2 \times (S_2 \wr S_2))$

Infinite fundamental domain - infinite number of escapees - lucky

$A = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$

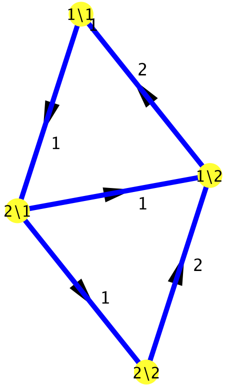


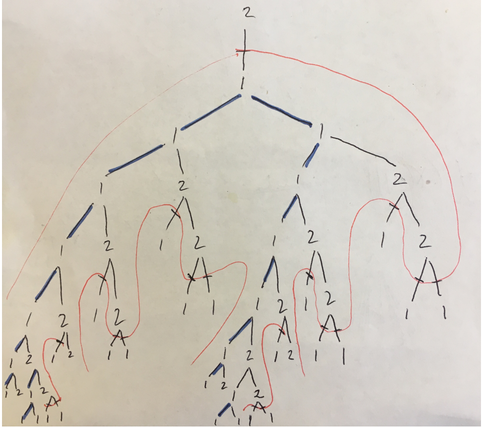


Fundamental domain is infinite with an infinite number of annoying rays. However, we can describe $2 \setminus 2$ in terms four $1 \setminus 1$s and an $S_2 \wr S_2$, and luckily $1 \setminus 1$ has a finite fundamental domain.
$\mathrm{Aut}(2 \setminus 2) \cong (\ldots \wr_{\partial FFD(1\setminus 1)} \{1\} \times S_2 \times (S_2 \wr S_2)) \wr S_2 \wr S_2$

Infinite fundamental domain - 2 escapees - lucky

$A = \left(\begin{smallmatrix} 2 & 1 \\ 2 & 1 \end{smallmatrix}\right)$

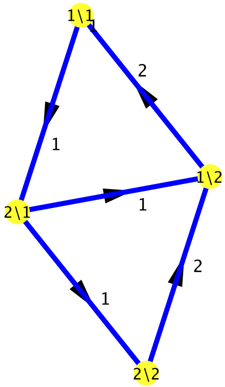


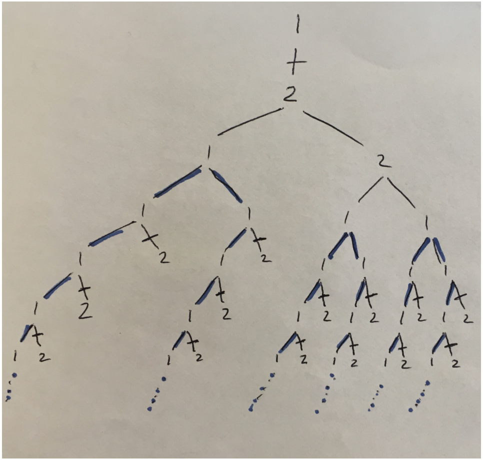


There is a trajectory that starts at $1 \setminus 2$ and never returns which has weight 2. The cycle part has weight 1. 2 annoying ends. $\mathrm{Aut}(1 \setminus 2) \cong \mathrm{Aut}(1 \setminus 1) \wr S_2 \cong (\ldots \wr_{\partial FFD(1 \setminus 1)} \{1\} \times S_2 \times (S_2 \wr S_2)) \wr S_2$ (from earlier)

Infinite fundamental domain - 6 escapees - lucky

$A = \left(\begin{smallmatrix} 2 & 1 \\ 2 & 1 \end{smallmatrix}\right)$

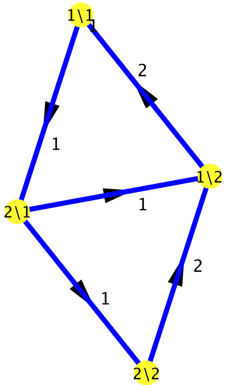


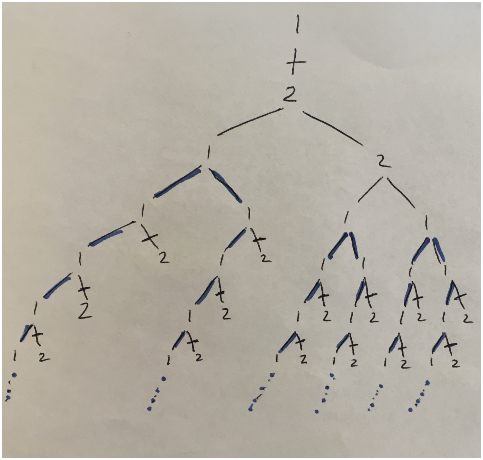


There are two trajectories that start at $2 \setminus 1$ and never return. One has weight two and the other has weight 4. The cycle part has weight 1. 6 annoying ends.

Infinite fundamental domain - 6 escapees - lucky

$A = \left(\begin{smallmatrix} 2 & 1 \\ 2 & 1 \end{smallmatrix}\right)$





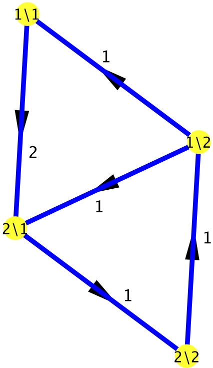
$\mathrm{Aut}(2 \setminus 1) \cong \wr_{\partial IFD}(S_2 \times (S_2 \wr S_2))$?

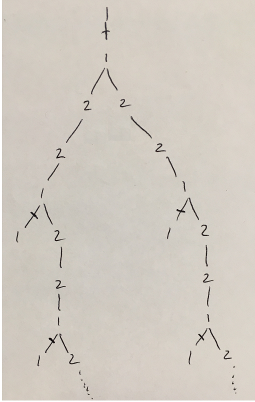
$\mathrm{Aut}(2 \setminus 1) \cong \mathrm{Aut}(1 \setminus 1) \wr (s_2 \times (S_2 \wr S_2)) \cong$

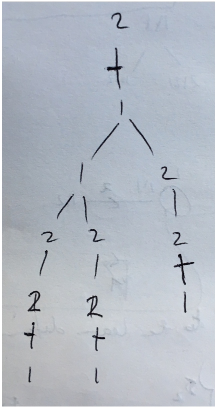
$(\ldots \wr_{\partial FFD(1 \setminus 1)} \{1\} \times S_2 \times (S_2 \wr S_2)) \wr (S_2 \times (S_2 \wr S_2))$

Infinite fundamental domain- 2 escapees - lucky

$A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$



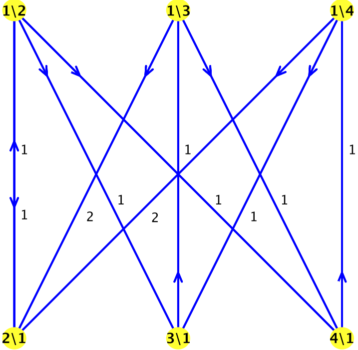


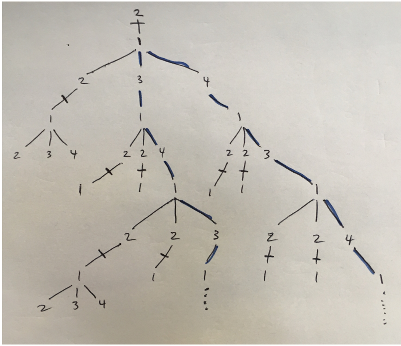


There is one trajectory that starts at $1 \setminus 1$ and never returns. It has weight 2. The cycle part has weight 1. 2 annoying ends.
$\mathrm{Aut}(1 \setminus 1) \cong (\mathrm{Aut}(1 \setminus 1) \times \{1\}) \wr S_2 \cong \ldots (S_2 \times \{1\}) \wr (S_2 \times \{1\}) \wr S_2$
or $\mathrm{Aut}(1 \setminus 1) \cong \mathrm{Aut}(1 \setminus 2) \wr S_2 \cong (\ldots \wr_{\partial FFD(1\setminus 2)} (S_2 \times \{1\})) \wr S_2$

Infinite fundamental domain - 2 escapees - unlucky

$$A = \begin{pmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}$$

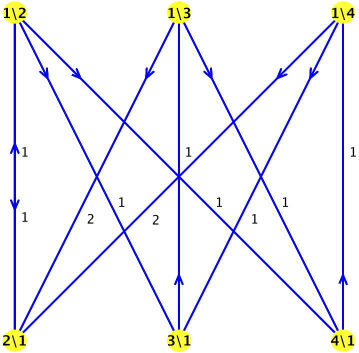


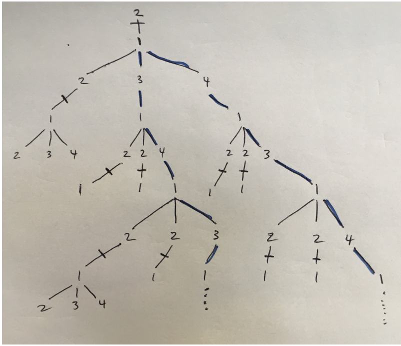


Two trajectories from $1 \setminus 2$ with weight 1 that never return. 2 annoying ends which can't be permuted.

Infinite fundamental domain - 2 escapees - unlucky

$$A = \begin{pmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}$$





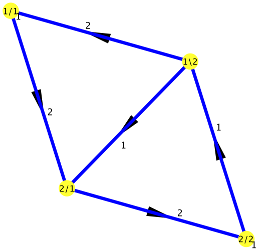
No single vertex removal destroys all cycles in $X(M)$. Would need to remove e.g. $1 \setminus 2$ and $1 \setminus 3$ and say

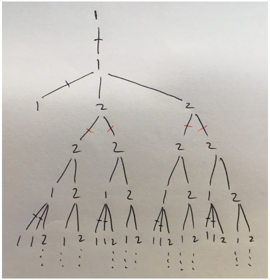
$\mathrm{Aut}(1 \setminus 2) \cong \mathrm{Aut}(1 \setminus 2) \times \mathrm{Aut}(1 \setminus 3) \times (\mathrm{Aut}(1 \setminus 2) \wr S_2) \times \mathrm{Aut}(1 \setminus 3)$,

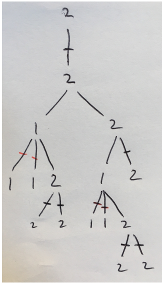
$\mathrm{Aut}(1 \setminus 3) \cong (\mathrm{Aut}(1 \setminus 2) \wr S_2) \times (\mathrm{Aut}(1 \setminus 2) \wr S_2) \times \mathrm{Aut}(1 \setminus 3)$.

Infinite fundamental domain - infinite number of escapees

$A = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}$







Note there are loops on $1 \setminus 1$ and $2 \setminus 2$. One of the basic $1 \setminus 1$-avoiding cycles has a weight of 2 and so there are an infinite number of annoying ends. Would need to remove $1 \setminus 1$ and $2 \setminus 2$.

$\mathrm{Aut}(1 \setminus 1) \cong \mathrm{Aut}(1 \setminus 1) \times (\mathrm{Aut}(2 \setminus 2) \wr S_2 \wr S_2)$

$\mathrm{Aut}(2 \setminus 2) \cong (\mathrm{Aut}(1 \setminus 1) \wr S_2) \times (\mathrm{Aut}(2 \setminus 2) \wr S_2) \times (\mathrm{Aut}(1 \setminus 1) \wr S_2) \times (\mathrm{Aut}(2 \setminus 2) \wr S_2) \times \mathrm{Aut}(2 \setminus 2)$

Other examples?

Can you have an infinite number of escapees per fundamental domain but none can be permuted? - I don't think so.

Can we have only one escapee per fundamental domain? - I don't think so.

Finite fundamental domain ($i \setminus j$ in terms of $i \setminus j$) iff all infinite paths in $X(M)$ starting at $i \setminus j$ revisit $i \setminus j$ iff (when $\|A\|_\infty > 2$) $i \setminus j$ is almost isomorphic to the forest of some number of $i \setminus j$s.

In the well-mixed case, this means all cycles in $X(M)$ pass through $i \setminus j$. Delete the $i \setminus j$ row and column from M (this is like deleting the vertex $i \setminus j$ from $X(M)$) and take higher and higher powers of this modified matrix, M', if this converges to 0 (M' is nilpotent, i.e. all eigenvalues are 0) then there are no $i \setminus j$-avoiding cycles in $X(M)$ and no annoying rays in $i \setminus j$ and so there is a finite fundamental domain of $i \setminus j$.

In not well-mixed case with $\|A\|_\infty > 2$, this means finite fundamental domain iff $i \setminus j$ is in the sink. In the case that $\|A\|_\infty = 2$ then $i \setminus j$ always has finite fundamental domain (but we aren't interested since the aut group of $i \setminus j$ will be trivial).

We can write $\mathrm{Aut}(i \setminus j)$ as a finite group wreath an infinitely iterated wreath product of a different finite group when we can write $i \setminus j$ in terms of $k \setminus l$, that is, when $i \setminus j$ is almost isomorphic to a forest of some number of $k \setminus l$. This happens when all infinite paths in $X(M)$ starting at $i \setminus j$ pass through $k \setminus l$ (there are no $k \setminus l$-avoiding paths). For the well-mixed case this means if $k \setminus l$ has a finite fundamental domain then $\mathrm{Aut}(i \setminus j)$ will be a finite group (the automorphism group of the 'cap' of $i \setminus j$) wreath an infinitely iterated wreath product of a different finite group ($\mathrm{Aut}(k \setminus l)$). For the not well-mixed case this means both are in the sink (but then both have finite fundamental domain anyway).

A subgroup of Aut($T_{d,r}$)?

Let the finite fundamental block of $k \setminus l$ have d children. Imagine shrinking each finite fundamental block into a single vertex, then the tree $k \setminus l$ would be a rooted tree T_d where each vertex has d children but only certain permutations of those children would be possible (those that are allowed by the automorphism group of the finite fundamental block). Therefore, Aut($k \setminus l$) is related to a subgroup of Aut(T_d). Also, Aut($i \setminus j$) is related to a subgroup of Aut($T_{d,r}$) (the rooted tree, where the root has r children and every other vertex has d children). What can be said when we don't have this?

Extreme examples

When there is only one cycle in $X(M)$, all the cycles pass through all the vertices and so every $i \setminus j$ has a finite fundamental domain.

When $A_{ii} \geq 2$, $i \setminus i$ in $X(M)$ has a loop. And so every other tree will fail to have a finite fundamental domain ($i \setminus i$ may have one though). If the main diagonal of A is filled with entries greater than or equal to 2, then no rooted trees will have a finite fundamental domain.

Level-homogeneous trees

Level-homogeneous trees are present in A iff there is a cycle of level-homogeneous trees in $X(M)$ that you get 'stuck' in. This is only possible if either A is not well-mixed and this cycle is the sink, or $\|A\|_\infty = 2$, or $\|A\|_\infty$ is well mixed and there is only one cycle in $X(M)$:

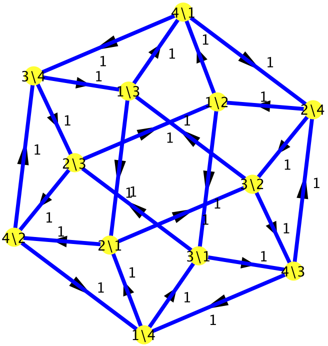
These include the regular and semi-regular trees as well as trees that look like these with subdivisions of edges done in a symmetric fashion, e.g. $A = \begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 1 \\ 0 & 3 & 0 \end{pmatrix}$, but you could also split each edge in a regular tree into $2k$ edges for any $k \in \mathbb{N}$ by adding $2k - 1$ more vertices, so that labelling along these edges in symmetric about the middle vertex.

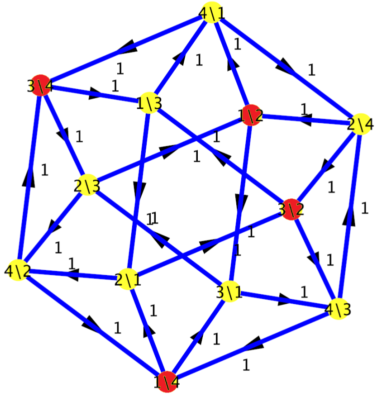
Are there examples of every cycle passes through a certain vertex in $X(M)$ but more than one cycle exists?

To answer “For a given A, is there some set of trees $B \subsetneq P_A$, such that the automorphism group of every tree in P_A can be written in terms of the automorphism groups of the trees in B?”

Need to talk about the period of an irreducible matrix. m/p is the size of each block, choose all vertices in any block and these will destroy all cycles. This doesn't help when M has period 1. What else can be said?
To delete vertices, in general the question is NP-complete
https://en.wikipedia.org/wiki/Feedback_vertex_set

$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$. To break all cycles you can delete $\{1 \setminus 2, 3 \setminus 2, 1 \setminus 4, 3 \setminus 4\}$. The period here is 1 (primitive).





Normal Subgroups?

Subgroup that only permute children of a certain label or labels (level at a time starting at the top)

Subgroups that only permute down to a certain level

Subgroups that only use certain subgroups of the symmetry group for each label.

Level-stabilisers and rigid-stabilisers (could also have restricition on permuting labels or subgroup of symmetry group of each label etc)

The automorphism group of a forest needs to account for permuting trees of the same type.

If $\mathcal{F}_x$ is a forest with forest vector x then

$$\mathrm{Aut}(\mathcal{F}_x) \cong \prod_{\{i \setminus j \in P_A \colon x_{i\setminus j} \neq 0\}} \mathrm{Aut}(i \setminus j) \wr S_{x_{i\setminus j}}$$